Direct Simulation for Wind Instrument Synthesis

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- Webster’s equation
- Finite difference schemes
- Efficiency, accuracy and stability
- Sound examples: Single reed wind instruments
**Webster’s Equation**

- Usual starting point for wind instrument models (and speech): an acoustic tube, surface area $S(x)$:

- Under various assumptions, velocity potential $\Psi(x,t)$ satisfies:

  $$\Psi_{tt} = c^2 \left(S \Psi_x\right)_x$$

- $\Psi(x,t)$ related to pressure $p(x,t)$ and volume velocity $u(x,t)$ by:

  $$p = \rho \Psi_t, \quad u = -S \Psi_x$$
Single Reed Model

- A standard one-mass reed model:

\[ \dot{y} + g \dot{y} + \omega_0^2 y + \omega_1^{1+\alpha} \left( [ -y + H ]^+ \right) \dot{y} = -a_1 p_\Delta \]

- Linear oscillator terms
- Collision term
- Driving term

- A driven oscillator:

\[ p_\Delta = p_m - p_{in} \]

Mouthpiece pressure drop

\[ u_m = a_2 [ -y + H ]^+ \sqrt{p_{\Delta} | \text{sgn}(p_{\Delta})} \]

Flow nonlinearity

\[ u_\text{in} = u_m - u_r \]

Flow conservation

\[ u_\text{in} = -S(0) \Psi_x (0, t) \]

Flow induced by reed

\[ p_{\text{in}} = \rho \Psi_\tau (0, t) \]

Bore coupling
Radiation Boundary Condition

- At the radiating end \((x=L)\), an approximate boundary condition is often given in impedance form:

\[
P(s) = Z(s)U(s) \quad \quad Z(s) = As - Bs^2
\]

- Models inertial mass and loss.
- BUT: not positive real \(\rightarrow\) not passive.
- A better approximation (p.r., passive):

\[
Z(s) = \frac{As}{1 + Bs/A}
\]

- When converted to the time domain:

\[
\Psi_x + q_1 \Psi_t + q_2 \Psi = 0 \quad \text{at} \quad x = L
\]
Finite Difference Scheme

- Sample bore profile at locations
  \( x = lh, \ l = 0, \ldots, N \)
  \( h = \) grid spacing

- Introduce grid function \( \Psi \), at locations
  \( x = lh, \ l = 0, \ldots, N \)
  \( t = nk, \ n = 0, \ldots \)
  \( k = \) time step

- Here is one particular finite difference scheme (explicit, 2\(^{nd}\) order accurate)

\[
\Psi_l^{n+1} = 2\lambda^2 \frac{S_l + S_{l+1}}{S_{l+1} + 2S_l + S_{l-1}} \Psi_l^n + 2\lambda^2 \frac{S_l + S_{l-1}}{S_{l+1} + 2S_l + S_{l-1}} \Psi_{l-1}^n + 2(1 - \lambda^2) \Psi_l^n - \Psi_{l-1}^{n-1}
\]

- Courant number \( \lambda \) defined as \( \lambda = ck/h \)
Stability and Special Forms

- Can show (energy methods) that scheme is stable, over interior, when $\lambda \leq 1$
- When $\lambda = 1$, scheme simplifies to:

$$
\Psi_l^{n+1} = 2 \frac{S_l + S_{l+1}}{S_{l+1} + 2S_l + S_{l-1}} \Psi_l^n + 2 \frac{S_l + S_{l-1}}{S_{l+1} + 2S_l + S_{l-1}} \Psi_{l-1}^n - \Psi_{l-1}^{n-1}
$$

...equivalent to Kelly-Lochbaum scattering method

- When $\lambda = 1$, and $S = \text{const.}$, scheme simplifies further:

$$
\Psi_l^{n+1} = \Psi_{l+1}^n + \Psi_{l-1}^n - \Psi_l^{n-1}
$$

...equivalent to digital waveguide (exact integrator)
Stability Condition and Tuning

- Stability condition requires $\lambda \leq 1 \rightarrow h \geq ck$
- For simplicity, would like to choose an $h$ which divides $L$ evenly, i.e.,
  $$L / h = N \quad \text{for integer } N$$
- Not possible for waveguide/Kelly-Lochbaum methods --- $h=ck$. Result: detuning, remedied using fractional delays.
- In an FD scheme, can choose $h$ as one wishes. Result: very minor dispersion/loss of audio bandwidth. Numerical cutoff:
  $$f_c = \frac{f_s}{\pi} \sin^{-1}(\lambda) \leq \frac{f_s}{2}$$
- Worst case near $f_s = 44.1$ kHz, typical wind instrument dimensions:
  $$f_c \approx 20 \text{ kHz}$$  
  $$f_s \approx 44037 \text{ Hz}$$  
  $$f_s \approx 44036 \text{ Hz}$$
Accuracy—Modal Frequencies

- Numerical dispersion---normally a problem for FD schemes!
- This is a 2\textsuperscript{nd} order scheme---might expect severe mode detunings…
- Not so…

E.g., for a lossless clarinet bore…

...calculated modal frequencies are nearly exact, over the entire spectrum

<table>
<thead>
<tr>
<th>Mode</th>
<th>Freq. (FD, Hz)</th>
<th>Freq. (exact, Hz)</th>
<th>cent diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>141.89</td>
<td>141.96</td>
<td>0.86</td>
</tr>
<tr>
<td>2</td>
<td>413.79</td>
<td>413.95</td>
<td>0.65</td>
</tr>
<tr>
<td>3</td>
<td>705.55</td>
<td>705.55</td>
<td>0.00</td>
</tr>
<tr>
<td>12</td>
<td>3144.04</td>
<td>3142.63</td>
<td>-0.77</td>
</tr>
</tbody>
</table>
Accuracy—Transfer Impedance

- Even under more realistic conditions (i.e., with radiation loss), behaviour is extremely good:
- Transfer impedance (mouth → radiating end):

![Graph showing transfer impedance](image)

- **Red**: exact (calculated at 400 kHz)
- **Green**: calculated at 44.1 kHz

- Upshot: FD approximation converges very rapidly…
- …“perceptually” exact, even at audio sample rate.
- No compelling reason to look for better schemes…
### Explicit Updating

- **Discretization of oscillator:**
  \[
  \ddot{y} + g\dot{y} + \omega_0^2 y + \omega_1^{1+\alpha} \left( -y - H \right)^+ = -a_1 p_\Delta
  \]

  Parameterized implicit discretization

  
  
  Implicit discretization

  
  Exact integrator possible for linear part of oscillator...

  
  Explicit update...

<table>
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<tr>
<th>Parameterized implicit discretization</th>
<th>Implicit discretization</th>
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<tbody>
<tr>
<td>[ y_{n+1} - 2y^n + y^{n-1} + \frac{gk}{2} \left( y_{n+1} - y^{n-1} \right) + \frac{\omega_0^2 k^2 \theta y^n}{2} + \frac{\omega_1^2 k^2 (1-\theta)}{2} \left( v_{n+1} + y^{n-1} \right) = \frac{\omega_0^2 k^2}{2} \left( v_{n+1} + y^{n-1} \right) \left( -y^n - H \right)^+ ]</td>
<td></td>
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<td>[ u_r^n = \frac{a_1}{2k} \left( y_{n+1} - y^{n-1} \right) ]</td>
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<td>Bore coupling</td>
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<td>[ p_{in}^n = \frac{\rho}{2k} (\psi_{n+1}^0 - \psi_{n-1}^0) ]</td>
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<tr>
<td>[ u_{in}^n = -\frac{S_0}{2\eta} \left( \psi_r^n - \psi_r^n \right) ]</td>
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- Implicit discretization \(\rightarrow\) excellent stability properties
- Unknowns always appear linearly...
Explicit Updating

- Can find a flow path in order to update all the state variables (sequentially)

- Similar to setting of “reflection-free port resistances” in linear WDF networks...
- ...but more general.
Note on Stability

- The scheme for the bore + bell termination, in isolation, is guaranteed stable.
- Situation more complicated when reed mechanism is connected.
- Consider system under transient conditions (input $p_m = 0$):

$$H(t) = H_{bore}(t) + H_{bell}(t) + H_{reed}(t) \leq H(0)$$

- System is dissipative $\rightarrow$ state bounded for any initial conditions.
- Under forced conditions, would like:

$$H(t) \leq H(0) + \int_{0}^{t} f(p_m) \, dt'$$

- Upshot: impossible to bound solutions of model system under forced conditions
- Best one can do: ensure energy balance is respected in FD scheme...
Computational Cost

For a given sample rate $f_s$, bore length $L$, and wave speed $c$, the computational requirements are:

- $2Lf_s/c$ units memory
- $4Lf_s^2/c \rightarrow 6Lf_s^2/c$ flops/sec.

...independent of bore profile. Reed/tonehole/bell calculations are $O(1)$ extra ops/memory per time step

Example: clarinet $\rightarrow$ 15 Mflops/sec., at $f_s = 44.1$ kHz

Not a lot by today’s standards…far faster than real time.
Toneholes

- Not difficult to add in tonehole models:

- Can add terms pointwise to Webster’s equation:

\[
\Psi_{tt} = c^2 (S\Psi_x)_x + \sum_{q=1}^{M} \delta(x - x^{(q)}) m^{(q)}
\]

\[
m^{(q)} = A\Psi_{tt}(x^{(q)}, t) + B\Psi(x^{(q)}, t) \quad q = 1, \ldots, M
\]

- In FD implementation, can be added anywhere along bore (Lagrange interpolation):
GUI: Matlab
Sound Examples

- Clarinet:
- Saxophone:
- Multiphonics/squeaks:
Conclusion

Disadvantages:
- Costs more to compute than a typical waveguide model (but still not much!)

Advantages:
- Bore modeling becomes trivial...
- More general extensions possible (NL wave propagation)
- Far more design freedom that, e.g., WG/WD methods