Energy-stable Modelling of Contacting Modal Objects With Piecewise-Linear Interaction Force

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September 2, 2008 / DAFx08, Helsinki
Overview

Motivation and scope

Background, situation, terms
   Modelling impact interaction using force laws
   Facts about differential equations with linear operators
   Systems with finite-dimensional state space

Modelling contact by modal description
   Principle idea
   Contact of finite-dimensional systems
   Discrete-time realization of contact condition
   Example implementation

Conclusions and outlook
Interactive realtime synthesis of sounds of contacting solid objects: dynamic modelling of *continuous* and interrupted contact? (In particular: rolling...)  

- Modal description successful for many cases of *one single* solid object — under clear conditions of mathematical theory...  

- Used approaches for force signals: impulsive force approximation for very short contact, ad-hoc assumptions for longer single contacts (e.g. cosine half-periods...), force profiles for continuous interaction on the basis of more or less heuristic ideas...  

- *But*: situations of *varying, continuous and interrupted* contact?  
  - E.g.: ball falling, bouncing, rolling  
  - Possible answer: models *including interaction forces*...
Motivation, specific scope

Modelling of rolling sound through contact perpendicular to plane of contact.

Figure: Reduction of rolling geometry to 1-dimensional impact.

Summary: necessity of a computational model of impact accounting for “microscopic” interaction behavior and macroscopic/long-term energy preservation.
Background – Modelling impact interaction using force laws

Different laws have been proposed for the interrelation of “global distances” and the interaction force acting on the involved objects, e.g. by Hertz or by Hunt and Crossley:

\[
f(x, \dot{x}) = \begin{cases} 
-kx^{\alpha} - \lambda x^{\alpha} \cdot \dot{x}, & x > 0 \\
0, & x \leq 0 
\end{cases}
\] (1)

- Non-linearity of scenario, therefore: discrete-time implementation by numerical approximation
- But: even small inexactnesses lead to uncontrolled changes of the energy stored in the system and may add up in long-term behavior.
  - (The system as a whole is necessarily non-linear, even when linearizing the first line of equation (1).)

Central ideas: piece-wise linear behaviour of interaction force, exact solution in each linear stage, appropriate “switching”.
Background – general principles

Description of the temporal behaviour of a system

\[
\dot{\vec{z}}(t) = A\vec{z}(t) + \vec{f}_{\text{ext}}(t),
\]

(2)

\(\vec{z}(t)\) state of the system in the “*state space*” vector space \(Z\) — thus *state “vector”,* \(A\) a *linear* operator defined on state space.

Remarks:

- The state space may be finite- or also infinite-dimensional.
- The demand on \(A\) to be linear is central for the term of “modes of a system” to make sense.
- Core idea of modal approach: simplify and solve equation (2) by expressing the state vector \(\vec{z}\) in a basis of (generalised) eigenvectors of \(A\).
  - E.g.: if \(\vec{z}\) is an eigenvector of \(A\) to eigenvalue \(d\), (2) reduces to \(\dot{\vec{z}}(t) = d \cdot \vec{z}(t)\) *with \(d\) scalar!* (no external forces, \(\vec{f}_{\text{ext}}(t) = 0\), homogeneous version).
Background – Finite-dimensional state space

Finite-dimensional state space: operator $A$ “is” matrix. General solution (homogeneous case)

$$\vec{z}(t) = e^{tA}\vec{z}(0)$$

(3)

then allows for an exact (up to resolution of finite computer architecture) computation in discrete-time algorithm by means of the “(state) transition matrix” $e^{t\Delta A}$ (to the time step $t_\Delta$).

- In this case the modal approach of finding the generalised eigenvectors of $A$ consists in finding a similarity transformation

$$N := V^{-1}AV$$

(4)

to Jordan canonical form, $V$ non-singular.

- In most practical cases Jordan canonical form is diagonal, i.e. $A = VDV^{-1}(\iff D = V^{-1}AV)$, $D$ diagonal matrix.

Transition matrix: $e^{t\Delta A} = Ve^{t\Delta D}V^{-1}$, with $e^{t\Delta D}$ also diagonal.
Background – Stiffness and friction matrices

Most common practical application: system of second order differential equations of the form

\[ M\ddot{x}(t) + K\dot{x}(t) + Cx(t) = \bar{f}^{\text{ext}}(t), \]  

(5)

\( M, K \) and \( C \) matrices representing inertia, “stiffness” and “friction”.

- Reduction: \( \tilde{z} := \begin{pmatrix} \dot{x} \\ \ddot{x} \end{pmatrix} , \ A := \begin{pmatrix} O & E \\ -K & -C \end{pmatrix} \)

- Most often derived from Newton’s laws for an idealised system of lumped masses or from Lagrange equations, but may also be derived from a spatially continuous system which has first been transformed directly by the modal approach and then simplified.

- Inertia matrix \( M \) is generally invertible (most often diagonal) and may therefore be omitted \((K \text{ and } C \text{ to } M^{-1}K \text{ and } M^{-1}C)\).
Modelling contact by modal description

Two systems of the described type of equation (2)

\[
\dot{\vec{z}}_1(t) = A_1 \vec{z}_1(t) + \vec{f}_{1ext} \quad \text{and} \quad \dot{\vec{z}}_2(t) = A_2 \vec{z}_2(t) + \vec{f}_{2ext}
\]  

(6)

interacting by means of some interaction force \( \vec{f}(\vec{z}_1, \vec{z}_2) \). For simplicity:

▶ no other external forces,
▶ “actio–reactio” principle holds: \( \vec{f}_1(= \vec{f}_{1ext}) = \vec{f} \) and \( \vec{f}_2 = -\vec{f} \).

**General idea:** If \( \vec{f} \) is a linear function of \( \vec{z}_1 \) and \( \vec{z}_2 \), the system of both masses during contact may again be written in the form

\[
\dot{\vec{z}}(t) = A\vec{z}(t)
\]

with a linear operator \( A \) on the state space

\( Z = Z_1 \times Z_2 \), “built from” \( A_1, A_2 \) and the linear expression of \( \vec{f} \).
Contact of finite-dimensional systems – example, point–mass “hammer”

Modal object as in equation (5) struck by a point-mass “hammer” (to keep the overall system possibly simple and demonstrate the general idea),

\[ m_h \ddot{x}_{n+1} = f. \]  

(7)

One contact point \( l_{\text{con}} \), contact modelled as a massless damped-spring connection:

\[ f = \begin{cases} 
  k_{\text{con}} (x_{l_{\text{con}}} - x_{n+1}) + c_{\text{con}} (\dot{x}_{l_{\text{con}}} - \dot{x}_{n+1}), & \text{for } x_{n+1} < x_{l_{\text{con}}} \\
  0, & \text{otherwise.} 
\end{cases} \]  

(8)
Contact of finite-dimensional systems – example, point–mass “hammer”

Accelerations on hammer and contact point ($f_{\text{con}} = -f$ according to the actio–reactio principle):

\[ \ddot{x}_{n+1} = \frac{k_{\text{con}}}{m_h} x_{l,\text{con}} - \frac{k_{\text{con}}}{m_h} x_{n+1} + \frac{c_{\text{con}}}{m_h} \dot{x}_{l,\text{con}} - \frac{c_{\text{con}}}{m_h} \dot{x}_{n+1} \]  \hfill (9)

and

\[ \ddot{x}_{l,\text{con}} = \frac{k_{\text{con}}}{m_{l,\text{con}}} x_{n+1} - \frac{k_{\text{con}}}{m_{l,\text{con}}} x_{l,\text{con}} + \frac{c_{\text{con}}}{m_{l,\text{con}}} \dot{x}_{n+1} - \frac{c_{\text{con}}}{m_{l,\text{con}}} \dot{x}_{l,\text{con}} + f_{l,\text{con}}^{(\text{int})} , \]  \hfill (10)

$f_{l,\text{con}}^{(\text{int})}$ internal force acting on $m_{l,\text{con}}$ inside the object.

Combine positions of the $n$ masses of the object with position of the hammer $x_{n+1}$ into $\dot{\mathbf{x}} := (x_1 \ldots x_{n+1})^T$:

\[ \dddot{\mathbf{x}}(t) = -K_{\text{con}} \dot{\mathbf{x}}(t) - C_{\text{con}} \ddot{\mathbf{x}}(t) + \ddot{\mathbf{f}}^{(\text{int})}(t), \]  \hfill (11)

with
Contact of finite-dimensional systems – example, point–mass “hammer”

\[
K_{\text{con}} := \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
\vdots & & 0 & \cdots & 0 \\
0 & \cdots & \cdots & 0 & -\frac{k_{\text{con}}}{m} \\
0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & \frac{k_{\text{con}}}{m} \\
0 & \cdots & 0 & \cdots & 0 & \frac{k_{\text{con}}}{m_h} \\
0 & \cdots & 0 & \cdots & 0 & \frac{k_{\text{con}}}{m_h}
\end{pmatrix}
\]

\[\rightarrow l_{\text{con}}\text{th row} \quad (12)\]

\[C_{\text{con}} \text{ analogous.}\]

Combine matrices \( K \) and \( C \) with \( K_{\text{con}} \) and \( C_{\text{con}} \) into

\[
\dot{K} := \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} + K_{\text{con}} \quad \text{and} \quad \dot{C} := \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} + C_{\text{con}},
\]

to gain the final equation of the entire system during contact as

\[
\ddot{x}(t) + \dot{K}\dot{x}(t) + \dot{C}\dot{x}(t) = 0.
\]
Modelling contact by modal description

Remarks:

- Presented procedure applicable in an analogous way
  - for modal objects given by abstract attributes of modal frequencies, decay times, and weights at the point of contact,
  - if the “hammer” is not as simple as a point mass, but describable by means of a finite-dimensional linear operator.

- The principal idea also holds for systems with finite-dimensional state spaces, but its application is then generally much more involved and no general “receipt” for concrete solution can be given.
Discrete-time realization

Main points:

- Update steps in each linear phase (with transition matrices) are exact because based on analytical solution; in particular energy-preserving.

- **Crucial point**: “phase switching” — Exact moments of transition between different linear phases will not be at sample steps.

Figure: Single objects at discrete sampled moments just before and after occurrence of contact.

How to switch between non-contact and contact?
Discrete-time realization

Unbreakable constraints (here):

- Overlap of objects *must not occur*.
- Inserting a “contact spring” in stretched state means erroneous insertion of energy into system.

Present solution: usage of appropriate ideal, massless, perfectly stiff, connecting element.

Figure: Energy-preserving update at start of contact
Steps of discrete-time algorithm:

- No contact phase: update separate objects with individual transition matrices
  - Check if contact starts. If yes: go back one step and set contact. Use “offset element” to preserve energy.
- Contact phase: update with transition matrix of system in contact.
  - Check if still in contact. If no: release contact. Loss of energy at cut of compressed spring, but never insertion of energy.
Example implementation: string struck by “hammer”

Figure: Snapshots of a string struck by a free mass “hammer”.

Example implementation: string struck by “hammer” – slow motion view
Example implementation: string struck by “hammer” – realtime view, gravity
Example implementation: string struck by “hammer” – continuous contact and energy behaviour
Conclusions and outlook

- An approach and technique has been developed for dynamic modelling of contact interaction, *under complete control of system energy*.
- The model has been implemented and runs at low computational load on standard hardware.
- The current realization is very simple but will be refined for more complex scenarios:
  - Contact force laws with several linear phases approximating laws s.a. Hertz’s or Hunt and Crossley’s, or stick–slip friction.
  - More complex scenarios than one point-mass.
- Currently offline diagonalization, therefore fixed contact points. Idea: direct online computation of transition matrix...
Thank you!