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# Surface Smoothing Based on a Sphere Shape Model

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**ABSTRACT**

*The smoothing of polyhedral surfaces is important for the analysis and visualization of three-dimensional medical images. Simple Gaussian surface smoothing has a well-known drawback of causing surface shrinkage. This has led to various signal processing algorithms for surface mesh filtering. In this paper, a new approach to surface mesh smoothing without shrinkage is presented. The approach is based on a representation of sphere shaped simplex meshes. The surface smoothing is formulated as a quadratic optimization problem, which is shown to be uniquely solvable. The cost function to be optimized penalizes deviations from the sphere shape while maintaining the overall shape of the original mesh. A linear time gradient descent algorithm for solving the optimization problem is derived. The surface smoothing achieved this way is fast, simple, reversible, and does not cause surface shrinkage. We demonstrate good properties of the algorithm by filtering synthetic surfaces as well as surfaces extracted from three-dimensional medical images. In addition, quantitative comparisons to competing algorithms are made. These further illustrate the advantages of the proposed sphere filtering algorithm.*

## 1. INTRODUCTION

Volumetric imaging techniques, such as magnetic resonance imaging and positron emission tomography, produce more and more accurate three-dimensional representations of human anatomy and physiology. Their use for clinical and research purposes has been exploded in scale in recent years which has made analysis, processing, and visualization of medical images increasingly important areas of study. The three-dimensional (3-D) nature of volumetric images creates challenges for their visualization. It is usual that biologically interesting surfaces are extracted from image volumes in order to represent important data to a medical expert or to perform further measurements based on images. The extraction of surfaces can be done, for example, by iso-surface algorithms [1] after suitable image processing steps. However, extracted polyhedral surface meshes are often noisy and contain a (too) large number of polygons and hence important surface details can be hindered behind noise. Therefore, it is necessary to smooth surface meshes before their visualization or further analysis.

Early approaches to surface smoothing comprised mostly of optimization methods such as in [2]. These algorithms were of the quadratic time complexity, which forbade their use for the processing of large surface meshes - meshes of hundreds of thousands or millions of vertices are not uncommon. On the other hand, simple Gaussian (or Laplacian) smoothing, which is of linear time, causes surface shrinkage. This was well demonstrated in [3], where a surface smoothing scheme which is of linear time and causes no surface shrinkage was suggested. The idea was to low pass filter surface signals comprising of the vertex coordinates of polyhedral meshes. More recent developments include anisotropic filtering of the surface normal fields [4, 5]. A state-of-the-art report of the mesh filtering is given in [6].

In this paper, we propose a simple surface smoothing algorithm that is of linear time, causes no shrinkage and, is reversible. The reversibility is important if surfaces are smoothed for more efficient lossless compression. The smoothing of simplex meshes [7] is considered instead of the smoothing of triangle meshes. Representing surfaces as simplex meshes allows to formulate an efficient smoothing algorithm that is even simpler than the one in [3]. The proposed *sphere filtering algorithm* is based on a novel and compact description of a sphere surface [8, 9]. The algorithm is first formulated as an optimization problem for analysis of its properties and then a linear time steepest descent procedure for solving the optimization problem is presented.

## 2. BACKGROUND

### 2.1. Simplex meshes

In this paper, surfaces are represented using simplex meshes [7] which are topological duals of triangle meshes [10]. A simplex mesh consists of a set $S = \{s_1, \dots, s_n\}$ of vertex coordinates and a graph $\mathcal{G} = (V, E)$ with $V = \{1, \dots, n\}$ modeling the neighborhood relations between vertices. Here all vertex coordinates $s_i = (x_i, y_i, z_i)^T \in \mathbb{R}^3$. In a simplex mesh all vertices have exactly three neighbors i.e. the graph $\mathcal{G}$ is 3-regular. The set of neighbors of the vertex $i$ is denoted by $N_i$. Simplex meshes are studied because their constant vertex connectivity helps in the design of shape models, such as the sphere shape model described in the next subsection. The topological duality between simplex and triangular meshes allows for conversion of a triangular mesh to a simplex mesh and vice versa. A detailed presentation of a procedure can be found in [11].

However, we note that the topological duality does not imply the existence of the unique geometric bijection between simplex and triangle meshes, cf. [7].

Our interest lies in modifying the set of vertex coordinates leaving the topological structure of the mesh described by the graph $\mathcal{G}$ unaltered. Therefore, the symbol $S$ is used for the mesh and the neighborhood structure of the mesh is assumed to be known.

### 2.2. The sphere shape model

The sphere shape model derives the conditions for a simplex mesh to perfectly represent a sphere [8, 9]. That is, all vertices are supposed to lie on the given sphere and the distance between two neighboring vertices is a constant. We begin by defining the concepts of the reference point and the surface centered mesh. The *reference point* of the mesh $S$

$$g_S = \frac{1}{n}\sum_{i=1}^{n} s_i. \tag{1}$$

The *surface centered mesh* is then $\hat{S} = \{\hat{s}_i\} = \{s_i - g_S : i = 1, \dots, n\}$. For all $i = 1, \dots, n$, a simplex mesh perfectly representing the sphere satisfies

$$\hat{s}_i = \alpha \sum_{j \in N_i} \hat{s}_j, \tag{2}$$

where the *shape parameter*

$$\alpha = [3\cos(2\arctan\frac{2\sqrt{\pi\sqrt{3}}}{3\sqrt{n}})]^{-1}. \tag{3}$$

Perfectly spherical simplex meshes do not exist in general and hence Eqs. (2) are not necessarily solvable. However, based on the sphere shape model in Eq. (2), one can measure how well a simplex mesh approximates the perfectly spherical mesh.

The advantage of this shape representation is that it is simple and utilizes only linear operations on vertex coordinates. For instance, elegant shape representations derived in [7] involve computations that are nonlinear in vertex coordinates.

### 2.3. Gaussian smoothing of simplex meshes

The Gaussian filtering, sometimes termed the Laplacian filtering, is a basic way to smooth simplex meshes. To filter the mesh $S^0 = \{s_i^0\}$ iterative filtering according to the formula

$$s_i^{m+1} = s^m + \lambda[\frac{1}{3}\sum_{j \in N_i} s_j^m - s_i^m] \tag{4}$$

is performed. Then, after $M$ iterations $S^M$ is selected as the filtered mesh. The parameter $\lambda \in (0, 1)$ controls the amount of smoothing per iteration.

## 3. SPHERE SMOOTHING ALGORITHM

### 3.1. Optimization problem

Our task is the following: Given a bad quality mesh $S^0$, find a smooth, good quality mesh $S^*$ that has approximately the same shape than $S^0$. To solve this problem, we formulate it as a quadratic optimization problem

$$S^* = \arg\min_S f(S|S^0). \tag{5}$$

The cost function is

$$f(S|S^0) = \sum_{i=1}^{n} ||s_i - s_i^0||^2 + \lambda||\hat{s}_i - \alpha \sum_{j \in N_i} \hat{s}_j||^2, \tag{6}$$

where $\hat{s}_i$ denotes a vertex in a surface centered mesh, the shape parameter $\alpha$ is as in Eq. (3), and $\lambda > 0$ is a parameter specifying the amount of smoothing. The first term on the right hand side (r.h.s) of (6) is for maintaining the shape of the original mesh $S^0$. The second term on r.h.s of (6) tries to make the mesh sphere shaped. Because locally very smooth meshes are preferred, the parameter $\lambda$ should have a high value.

We derive now a closed form solution for (5). Denote $s_i = (x_i, y_i, z_i)^T$, $s_i^0 = (x_i^0, y_i^0, z_i^0)^T$. Let $x = (x_1, \dots, x_n)$, $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$ similarly for $y$ and $z$ coordinates. We can write $f(S|S^0) = f(x|x^0) + f(y|y^0) + f(z|z^0)$ with the obvious notation. It suffices to consider only $x$ coordinates of vertices when deriving solution to the minimization problem because solutions for other coordinates are equivalent.

Note that $\hat{x} = x - \frac{1}{n}\mathbf{1}x$, where $\mathbf{1}$ denotes the $n \times n$ matrix whose elements are all equal to one. Further, we define the symmetric $n \times n$ matrix $P = (p_{ij})$ as

$$p_{ij} = \begin{cases} 1 & \text{if} \quad j \in N_i \\ 0 & \text{otherwise} \end{cases}.$$

so that $\sum_{i=1}^{n} ||\hat{x}_i - \alpha \sum_{j \in N_i} \hat{x}_j||^2 = ||(I - \alpha P)\hat{x}||^2$. We can now write

$$f(x|x^0) = x^T Q x - 2{x^0}^T x + {x^0}^T x^0, \tag{7}$$

where

$$\begin{aligned} Q &= I + \lambda[(I - \alpha P)(I - \frac{1}{n}\mathbf{1})]^2 && (8) \\ &= I + \lambda(I - \alpha P + \frac{(3\alpha - 1)}{n}\mathbf{1})^2 && (9) \\ &= I + \lambda[(I - \alpha P)^2 - \frac{(3\alpha - 1)^2}{n}\mathbf{1}]. && (10) \end{aligned}$$

In above, we have repeatedly used the identity $P\mathbf{1} = \mathbf{1}P = 3\mathbf{1}$. Matrices are squared rather than multiplied with their transpose for brevity, and since all matrices involved are symmetric this notation is correct.

From Eq. (9), it is imminent that the matrix $Q$ is positive definite as a weighted sum of a positive definite matrix and a positive semidefinite matrix. Hence, the optimization problem (5) has the unique solution given by

$$x^* = Q^{-1}x^0, y^* = Q^{-1}y^0, z^* = Q^{-1}z^0. \tag{11}$$

### 3.2. Gradient descent algorithm

The inversion of the matrix $Q$ is not computationally feasible if processing large meshes. To solve (5) in linear time, a gradient descent algorithm is derived. Starting from $x^0$, each iteration of the algorithm is

$$x^{k+1} = x^k - \eta \nabla f(x^k|x^0) = x^k - 2\eta[Qx^k - x^0], \quad (12)$$

where the learning rate $\eta$ is a positive constant. The same formula is used for updating $y$ and $z$ coordinates. The algorithm is terminated when the maximum change in coordinates between iterations is below a defined threshold $\theta$, here we set $\theta \doteq (0.0001/n) \sum_i ||\hat{s}_i^0||$. We have experimentally found that values of $\eta$ that are below $\frac{1}{2.2\lambda}$ do not cause the algorithm to diverge. A rigorous convergence proof is still under construction.

Each step of the algorithm described by Eq. (12) is in linear time with respect to $n$. This is because multiplying a vector with the matrix $Q$ is a linear time operation as can be seen from (9): Matrices $I$ and $P$ are sparse and hence multiplication of a vector with them is linear time as is computing $\mathbf{1}x = (\sum_i x_i, \ldots, \sum_i x_i)^T$. Hence, calculating $Qx$ involves a constant number of linear time operations totaling a linear time operation.

## 4. EXPERIMENTS AND RESULTS

The parameter values were $\lambda = 400$, and $\eta = 1/(2.2\lambda)$ in all experients with the sphere filtering algorithm. In the first experiment, a surface mesh corrupted with multiplicative noise was filtered with the sphere, the Gaussian and the Taubin's $\lambda|\mu$ low-pass filtering algorithms. The noisy surface (in Fig. 1 (a)) was generated by adding multiplicative Gaussian noise to one half of the vertex coordinates of the spherical surface of radius of 10. The filtering results with the sphere and the Gaussian filters ($\lambda = 0.5$ and 50 iterations of Eq. (4)) are shown in Fig. 1. The sphere filter caused no surface shrinkage whereas the surface shrinkage caused by the Gaussian filter was considerable. Quantitative difference mesaures between the original (sphere without noise) surface and the filtered surfaces were also computed. The used criterion was the mean of squared differences between vertex positions of the original and the filtered mesh, abbreviated as MSE. The MSE between the original and noisy mesh was 1.86 and the MSE between the original and the sphere filtered noisy mesh was 0.05. (The exact solution by Eq. (11) yielded the MSE value 0.04.) The Gaussian filtering (with $\lambda = 0.5$) achieved the best MSE of 0.12 after 9 iterations; after 50 iterations the MSE was 0.90. We also tested Taubin's $\lambda|\mu$ low-pass filtering algorithm [3]. The best MSE, using the optimal parameters suggested by Taubin, was 0.13 and achieved after 90 iterations of the algorithm.

In Fig. 2 (a), a surface mesh generated by an iso-surface algorithm from $64 \times 64 \times 32$ synthetic image is shown. The quality of the mesh was poor due to a poor image resolution. In Fig. 2 (b), the sphere filtering result of the surface shown in Fig. 2 (a) is shown. As can be seen, the quality of the filtered mesh is much better than the quality of the original mesh.

In Fig. 3 (a), an inner surface of cerebral cortex extracted from a magnetic resonance image is shown (cf. [12] for the extraction procedure). The sphere filtering of the surface in Fig. 3 (a) is shown in Fig. 3 (b). Surfaces of cerebral cortex are known to be convoluted and as can be seen from Fig. 3, the sphere filtering algorithm preserved well the complex shape of the surface while surpressing the noise. The shape preservation is important because extracted cortical surfaces are used for quantitative statistical shape analysis of the human cerebral cortex, cf. [13].

The number of iterations of Eq. (12) required by the sphere filtering algorithm for these experiments varied between 200 and 500. The computation times of the algorithm implemented in Matlab (Mathworks, Natick, MA, US) varied between 0.5 seconds in the noisy sphere experiment (Fig. 1) to 1 minute in the cortical surface experiment (Fig. 3). The number of vertices in meshes were 1280 and 81920, respectively.

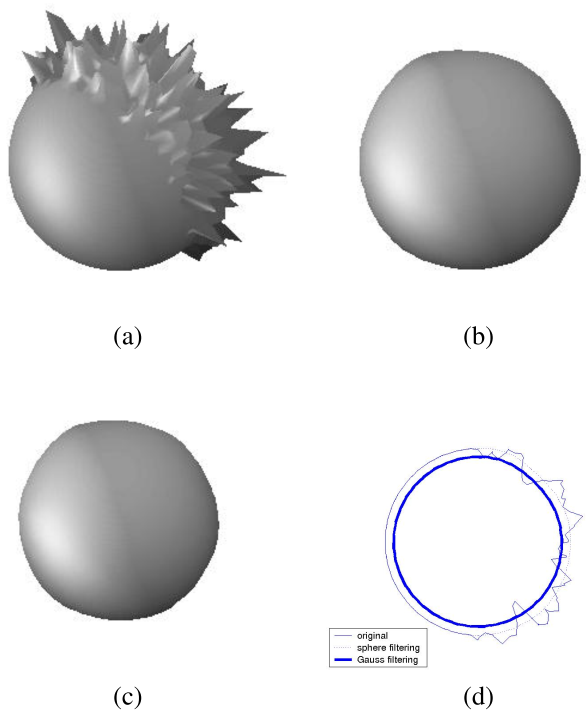
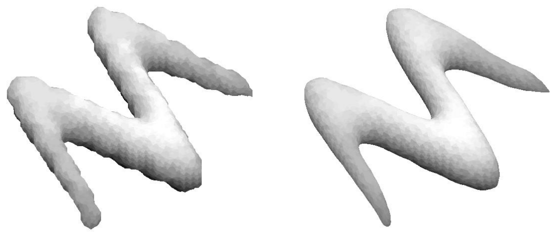



**Fig. 1:** Surface filtering results. (a) A noisy surface to be filtered. (b) Smoothing result using sphere filtering algorithm with $\lambda = 400$. (c) Result using a Gaussian filter. (d) A cross-section of the noisy surface, sphere filtered surface, and Gaussian filtered surface. The sphere filter causes no shrinkage whereas the Gaussian filter clearly shrinks the surface.

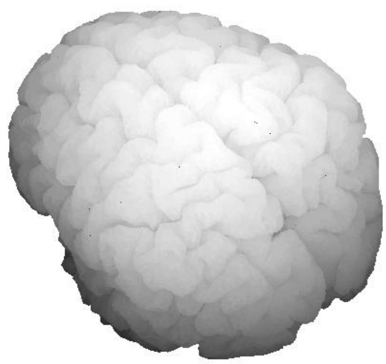
## 5. DISCUSSION AND SUMMARY

We have proposed the sphere filtering algorithm for the smoothing of simplex meshes. The algorithm has many favorable properties: it causes no surface shrinkage, it is of linear time, and it is extremely simple. We have demonstared the good properties of the algorithm by filtering synthetic surface meshes and meshes extracted from

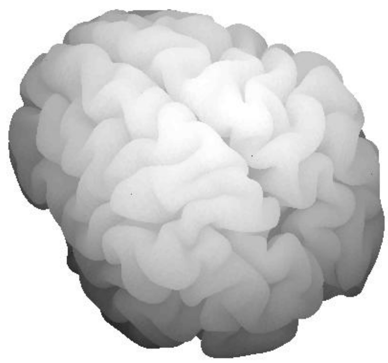


(a) (b)

**Fig. 2:** (a) A surface mesh generated by iso-surface algorithm and (b) sphere filtering of the mesh (a). Surfaces are flat-shaded to enhance the faceting effect.



(a)



(b)

**Fig. 3:** (a) A surface representing the inner surface of cerebral cortex. (b) The sphere filtered cortical surface. Surfaces are flat-shaded to enchance the faceting effect.

medical images. Moreover, the quantitative results obtained by the sphere filtering algorithm were superior to the ones obtained by competing algorithms. A drawback of the algorithm is that it requires surfaces to be represented by simplex meshes. Whereas simplex meshes are widely applied by medial image analysis community, their visualization is somewhat more challenging than the visualization of triangle meshes (see [7] for discussion). However, in our experience, this presents no real problem in practise because if represented by a rich enough data structure, simplex meshes can be converted to triangular ones in linear time and vice versa.

## 6. ACKNOWLEDGMENTS

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