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Management of Uncertainty within Estimation in Dynamical Context

Application to MEMS

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ABSTRACT

A probabilistic approach is proposed to manage uncertainty when dealing with estimation within dynamical context. The approach, which utilizes a time-space separation technique, relies on McShane's theory. The starting point is a (knowledge-based) model of the system where the estimation problem is set. The method was proposed in a general multidimensional case [2]; it is explained here by the light of an engineering application.

1. INTRODUCTION

A "measurement" is any quantity to be observed within a system. We talk about indirect measurement when the observation cannot be directly performed by some sensors. "Indirect measurement" raises in a variety of applications and leads to the paramount estimation theory. The starting point in the resolution of such an estimation problem is the model- a mathematical description of the system where the problem of indirect measurement is set. In some model, the quantities that when fixed cause the others to be uniquely determined are called "model's data" such as the imposed conditions on the solution of an ordinary or partial differential equation, observations, controls, parameters, etc. Often some of the model's data are unknown, this is expressed by the term "uncertainty". Prior information about some unknown can be inquired. It will consist of statistics that approximate some of its moments, if it is random or of some set where it takes its values if it is deterministic. In such a situation, the estimation has to come face to face with the propagation of the information from the unknown model's data to the quantity of interest- the measurement. The management of uncertainty within dynamical systems implies two aspects: mathematical tools for stochastic processes calculus and a procedure to construct models of these systems. McShane has undertaken these issues and has elaborated an accessible and unified theory [1], used here to model the relation that binds the measurement to the uncertainty. Finally, it is worthwhile noting that the term "dynamical" refers to the evolution of the measurement in time in one hand, and means that the model comprises at least one dynamical relation- a time differentiation or integration, in the other hand.

This paper is organized as follows: In section 2, the modeling task is described in general. It is based on McShane's theory of stochastic differential equations (SDE). In section 3, illustration of this task on an engineering application is given, and section 4 concludes the paper.

2. MODELING

Let us consider an indirect measurement problem as described in the introduction; naturally, the measurement m must be well localized before. First the initial model (that is available or is to be established) is transformed into the following canonical form- a first order differential system together with the expression of the measurement as a function of the state:

$$\begin{aligned} \dot{x}_i &= \mathrm{f}(t,x) + \sum_{\rho=1}^{r} \mathrm{g}^i_\rho(t,x) v_\rho \quad i = 1,\dots,n \\ m &= \psi(t,x) \end{aligned} \tag{1}$$

The model under the canonical form (1) is built so as to have (refer to [2] for the justification):

- $v_\rho(t,\omega)$, ρ=1,...,r, scalar random functions of time, statistically independent,
- $x_i(0,\omega)$, i=1,...,n, random variables independent of $v_\rho(t,\omega)$, ρ=1,...,r, and having known joint probability distribution.

The system of equations

$$\dot{x}_i = \mathrm{f}(t,x) + \sum_{\rho=1}^{r} \mathrm{g}^i_\rho(t,x) v_\rho \quad i = 1,\dots,n \tag{2}$$

is reminiscent of the state equation in control theory, that's why x is called state of the model under the canonical form. The reason why it is to be this form, is the fact that from the model under the canonical form a model under the form of a SDE will be derived, and this ultimate SDE is exactly the integral form of the equations in (2) where v_ρ, ρ=1,...,r, are idealized to white noises, of course via some **correction**. This derivation is based on McShane's theory [1]:

$$dx_i = \mathrm{f}(t,x)dt + \sum_{\rho=1}^{r} \mathrm{g}_\rho^i(t,x)dw + \sum_{\rho,\xi=1}^{r} \mathrm{h}_{\rho,\xi}^i(t,x)dw_\rho dw_\xi \qquad (3)$$

where

$$\mathrm{h}_{\rho,\xi}^i(t,x) = \frac{1}{2}\sum_{k=1}^{n} \frac{\partial\, \mathrm{g}_\rho^i}{\partial x_k}(t,x)\, \mathrm{g}_\xi^i(t,x) \qquad (4)$$

If the particular SDE (3) is defined (i.e. the conditions for the existence of solutions, and for the solution to be unique within a SDE in general [1] are satisfied) for a large family of integrator processes w_ρ, ρ=1,...,r, including Lipschitzian processes and Brownian motion processes, it is said that the model (3) has the property of inclusiveness.

It is said that the model (3) has the property of consistency if the solution of the SDE calculated with w_ρ, ρ=1,...,r, Lipschitzian processes agrees with that calculated with w_ρs Brownian motion processes. In fact, Lipschitzian processes are more realistic for modeling, whereas Brownian motion processes are more manageable in calculi but do not exist in reality. So we need these two properties to legitimize the use of Brownian motion processes in further calculi (such as Fokker-Planck-Kolmogorov (FPK) equation establishment, etc), and to guarantee that the obtained result applies also to the (trustworthy) model (i.e. having Lipschitzian integrator processes).

Actually, the model (3-4) has the desired properties of inclusiveness and consistency provided the following (necessary and sufficient) condition [1]:

$$\sum_{k=1}^{n} \frac{\partial\, \mathrm{g}_\rho^i}{\partial x_k}(t,x)\, \mathrm{g}_\xi^i(t,x) = \sum_{k=1}^{n} \frac{\partial\, \mathrm{g}_\xi^i}{\partial x_k}(t,x)\, \mathrm{g}_\rho^i(t,x) \quad \forall \rho,\xi,n$$

The final step in the modeling is "to go back to" the measurement. To do this, we construct a vector *me* called extended measurement such that the mapping $me=\psi(x)$ would be an homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$. McShane's chain rule gives the SDE of *me* from that of x.

3. ILLUSTRATION

Here is an application to illustrate all the ideas of the above section on modeling. It's about the robust design for a microaccelerometer, as regards to the uncertainties in the fabrication process (manufacturing tolerances and errors). A typical microaccelerometer is composed of a vibrating plate, supporting beams and electrodes for driving and sensing. It is assumed that the vibrating plate oscillates only in one direction-the lateral (x-axis) which is both the driving and sensing direction. When an acceleration is applied (which is to be measured by the accelerometer), the beams which are about the y-axis flex; their deformation $\mathrm{d}(t, y)$ is described by the following model.

3.1 Knowledge-Based Model

$$\rho\, h \frac{\partial^2 \mathrm{d}}{\partial t^2}(t,y) + \frac{\gamma h^3}{12(1-\nu^2)} \frac{\partial^4 \mathrm{d}}{\partial y^4}(t,y) = 0 \quad 0 \le y \le l \qquad (5)$$

$$\mathrm{d}(0,y) = \frac{\partial \mathrm{d}}{\partial t}(0,y) = 0 \qquad (6)$$

$$\mathrm{d}(t,0) = \frac{\partial \mathrm{d}}{\partial t}(t,0) = 0 \qquad (7)$$

$$\frac{\partial \mathrm{d}}{\partial x}(t,l) = \frac{\partial^3 \mathrm{d}}{\partial x^3}(t,l) = 0 \qquad (8)$$

l is the length of the beam; it's about the y-axis, and h is its thickness. γ, ν are Young's modulus of elasticity and Poisson's ratio respectively.

The effect of manufacturing tolerances and errors in a microelectromechanical system (MEMS) is more significant than in a macro-scaled one because a MEMS has a large error-to-size ratio. Thus a robust design of a MEMS passes through the study of such an effect. Here we assume that uncertainty in the fabrication process concerns only the thickness h of a beam:

$$h_{\inf} \le h \le h_{\sup} \qquad (9)$$

The effect of the uncertain model's data h, on the response $\mathrm{d}(t,l)$ (the plate displacement) is to be estimated, so it is the measurement (denoted by m) as settled in the introduction.

3.2 Model under the Canonical Form

The objective is to transform the model (5-9) into the canonical form (1). Consider the transformation

$$\begin{aligned} \mathrm{T}\{\mathrm{d}(t,y)\} &= \mathrm{D}(t) \\ &= \int_0^l \mathrm{d}(t,y)\, \mathrm{K}(y)\, dy \end{aligned} \qquad (10)$$

To apply T on (5), we have to calculate

$$\int_0^l \mathrm{O}(\mathrm{d})(t,y)\,\mathrm{K}(y)\,dy$$

where

$$\mathrm{O}(\mathrm{d})(t,y) = \frac{\partial^4 \mathrm{d}}{\partial y^4}(t,y)$$

Integration by parts gives

$$\int_0^l \mathrm{O}(\mathrm{d})(t,y)\,\mathrm{K}(y)\,dy = \left[\frac{\partial^3 \mathrm{d}}{\partial y^3}\mathrm{K}\right]_0^l - \left[\frac{\partial^2 \mathrm{d}}{\partial y^2}\frac{d\,\mathrm{K}}{dy}\right]_0^l + \left[\frac{\partial \mathrm{d}}{\partial y}\frac{d^2\,\mathrm{K}}{dy^2}\right]_0^l - \left[\mathrm{d}\frac{d^3\,\mathrm{K}}{dy^3}\right]_0^l + \int_0^l \mathrm{d}(t,y)\,\mathrm{O}(\mathrm{K})(y)\,dy \quad (11)$$

Let K be such that

$$\mathrm{O}(\mathrm{K})(y) = -\beta^2\,\mathrm{K}(y) \quad (12)$$

Regarding the imposed conditions on d in (6-7-8), if K has the following imposed conditions:

$$\mathrm{K}(0)=0 \quad \left.\frac{d\,\mathrm{K}(y)}{dy}\right|_{y=0} = \left.\frac{d\,\mathrm{K}(y)}{dy}\right|_{y=l} = 0 \quad \left.\frac{d^3\,\mathrm{K}(y)}{dy^3}\right|_{y=l} = 0 \quad (13)$$

then (11) gives

$$\int_0^l \mathrm{O}(\mathrm{d})(t,y)\,\mathrm{K}(y)\,dy = -\beta^2 \int_0^l \mathrm{d}(t,y)\,\mathrm{K}(y)\,dy$$

So when T is applied on (5), we get

$$\rho h \frac{d^2\,\mathrm{D}(t)}{dt^2} - \frac{\beta^2 \gamma h^3}{12(1-\nu^2)}\mathrm{D}(t) = 0 \quad (14)$$

Equations in (12-13) form a Sturm-Liouville problem, where K is an eigenfunction of the operator O and β is the corresponding eigenvalue. In the following a solution of the Sturm-Liouville problem (11-12) is sought.

The general solution of (12) is

$$\mathrm{K}(y) = a\exp\left(\frac{\sqrt{\beta}(1+j)}{\sqrt{2}}y\right) + b\exp\left(-\frac{\sqrt{\beta}(1+j)}{\sqrt{2}}y\right) + c\exp\left(\frac{\sqrt{\beta}(1-j)}{\sqrt{2}}y\right) + d\exp\left(-\frac{\sqrt{\beta}(1+j)}{\sqrt{2}}y\right)$$

where a, b, c, d are constants and j is the square root of -1.

K(0)=0 implies that $a+b+c+d=0$.

$\left.\frac{d\,\mathrm{K}(y)}{dy}\right|_{y=0} = 0$ implies that $a-b+c-d=0$ and $a-b-c+d=0$.

So $a=b$, $c=d$, $a=-c$, and

$$\mathrm{K}(y) = j4a\,\sin\left(\sqrt{\frac{\beta}{2}}y\right)\sinh\left(\sqrt{\frac{\beta}{2}}y\right)$$

$\left.\frac{d\,\mathrm{K}(y)}{dy}\right|_{y=l} = 0$ implies that β is the solution of the following equation

$$\tanh\left(\frac{l\sqrt{\beta}}{\sqrt{2}}\right) = -\tan\left(\frac{l\sqrt{\beta}}{\sqrt{2}}\right)$$

Let E_1 denote the set of points in $\mathbb{R}_+$ where $-\tan(x)$ and $\tanh(x)$ intercept.

$\left.\frac{d^3\,\mathrm{K}(y)}{dy^3}\right|_{y=l} = 0$ implies that β is the solution of the following equation

$$\tanh\left(\frac{l\sqrt{\beta}}{\sqrt{2}}\right) = \tan\left(\frac{l\sqrt{\beta}}{\sqrt{2}}\right)$$

Let E_2 denote the set of points in $\mathbb{R}_+$ where $\tan(x)$ and $\tanh(x)$ intercept. Finally, consider $E=E_1 \cup E_2$ where points are set in increasing order, the solution of the Sturm-Liouville problem is the following set of eigenfunction-eigenvalue couples:

$$\begin{cases} \mathrm{K}_i(y) = \mathrm{K}_0 \sin\left(\sqrt{\frac{\beta_i}{2}}y\right)\sinh\left(\sqrt{\frac{\beta_i}{2}}y\right) \\ \beta_i = 2\left(\frac{x_i}{l}\right)^2 \quad x_i \in E \quad i = 1,2,3... \end{cases} \quad (15)$$

where K_0 does not depend on y. Now let's proof that $\{\mathrm{K}_i\}$ is a family of orthogonal functions with respect to the scalar product

$$(f,g) = \int_0^l f(y)g(y)dy$$

For $i \neq i'$, integration by parts together with (13) imply that

$$\int_0^l \mathrm{O}(\mathrm{K}_i)(y)\,\mathrm{K}_{i'}(y)\,dy = \int_0^l \mathrm{K}_i\,\mathrm{O}(\mathrm{K}_{i'})(y)\,dy$$

So

$$\left(\beta_i - \beta_{i'}\right)\int_0^l \mathrm{K}_i(y)\,\mathrm{K}_{i'}(y)\,dy = 0$$

As the eigenvalues are distinct we deduce the desired property.

Any piecewise continuous function on a bounded domain can be expressed as a Fourier series, on the base of a family of orthogonal functions. So the deformation can be written as

$$\mathrm{d}(t,y) = \sum_{i=1}^{\infty} \varphi_i(t)\,\mathrm{K}_i(y) \tag{16}$$

It is easy to see that

$$\varphi_i(t) = \frac{\mathrm{D}_i(t)}{\int_0^l \mathrm{K}_i^{\,2}(y)dy} \tag{17}$$

where D_i is defined by (10). If we denote

$$\widetilde{\mathrm{D}}_i(t) = \frac{K_0}{\int_0^l \mathrm{K}_i^{\,2}(y)dy}\mathrm{D}_i(t)$$

then

$$\mathrm{d}(t,y) = \sum_{i=1}^{\infty} \widetilde{\mathrm{D}}_i(t)\sin\left(\sqrt{\frac{\beta_i}{2}}y\right)\sinh\left(\sqrt{\frac{\beta_i}{2}}y\right) \tag{18}$$

It is worth noting that (18) constitutes the inverse transformation of T. We truncate the series in (18), say at the order i=3, we write (14) for $\widetilde{\mathrm{D}}_i(t)$, i=1, 2, 3 and we set

$$x_1 = \widetilde{\mathrm{D}}_1(t) \quad x_2 = \frac{d\,\widetilde{\mathrm{D}}_1(t)}{dt} \quad x_3 = \widetilde{\mathrm{D}}_2(t) \quad x_4 = \frac{d\,\widetilde{\mathrm{D}}_2(t)}{dt}$$
$$x_5 = \widetilde{D}_3(t) \quad x_6 = \frac{d\widetilde{D}_3(t)}{dt}$$

the following first order differential system is obtained

$$\begin{cases} \dot{x}_1 = x_2 \quad \dot{x}_2 = \dfrac{\beta_1^{\,2}\gamma h^2}{12\rho(1-\nu^2)}x_1 \quad \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{\beta_2^{\,2}\gamma h^2}{12\rho(1-\nu^2)}x_4 \quad \dot{x}_5 = x_6 \quad \dot{x}_6 = \dfrac{\beta_3^{\,2}\gamma h^2}{12\rho(1-\nu^2)}x_5 \end{cases} \tag{19}$$

It is easy to see that

$$m(t) = \sum_{i=1}^{3} \widetilde{\mathrm{D}}_i(t)\sin\left(\sqrt{\frac{\beta_i}{2}}l\right)\sinh\left(\sqrt{\frac{\beta_i}{2}}l\right) \tag{20}$$

The initial conditions (6) imply that

$$x_i(0) = 0 \quad i = 1,2,3 \tag{21}$$

So the model under the canonical form is complete (19-20-21).

Then we derive the model under the form of a SDE for the extended measurement *me* as described, in general, in the section 2. In front of such a model, the original problem (the indirect measurement) which is formulated as the estimation of the measurement *m*, is reformulated as the estimation of the probability density function (pdf) of the extended measurement *me*. The pdf of *m* is deduced by marginalization as it is a component of *me*. Numerous methods for pdf estimation have been elaborated in [2]: Monte-Carlo's method, numerical resolution of the FPK equation, a method using MCMC [3], and a method using a generalized expectation [4].

4. CONCLUSION

A method to manage the uncertainty within estimation in dynamical context is proposed when illustrated on an engineering application. The modeling is based on one hand on McShane's theory of stochastic calculus, and on the other hand on an original operational technique (through a Sturm-Liouville formulation). It is worth noting that the transformation into a model under the canonical form (1) is crucial within our method. The proposed operational technique does not apply universally, but is often likely to work. In this particular application, we are in front of an infinite dimensional model due to the Fourier series involved; fortunately, the truncation we are led to can be controlled (in order and error). Finally, the fact that the obtained model is linear does not affect the generality of our method.

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