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An Extension of Bussgang's Theory to Complex-Valued Signals

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ABSTRACT

The paper presents theoretical results related to an extension of Bussgang's theory to complex-valued signals. We prove a few theorems that allow Bussgang's theory to be generalised to functions of complex variables. The derived results can be useful when analysing signal-processing systems with nonlinear devices fed by complex-valued Gaussian signals.

1. INTRODUCTION

Price's theorem [1] presented in 1958 and its extension given by E.L. McMahon [2] , A. Papoulis [3] and J.L. Brown [4] have been widely employed in different areas of engineering for the analysis of passing real Gaussian signals through nonlinear devices. A very important and widely used in practice corollary of Price's theorem is Bussgans's theorem [5] which states that if a real Gaussian signal passes through a memoryless nonlinear device then the output-input cross correlation function is proportional to the input autocovariance. In turn, from Bussgang's theorem a linear representation of the output of a memoryless nonlinearity directly follows. According to the theorem the nonlinearly distorted signal can be represented as a scaled input signal corrupted by nonlinear noise.

Modern system theory extensively employs the complex representation of the signals in the form of their complex envelopes [5]. For example, the use of complex-valued signals in modern communication theory is caused by the application of spectrally and power effective modulation formats.

At the same time, composite signals are widely used in modern signal-processing systems. In communication, such are orthogonal frequency division (OFDM) or code division multiple access (CDMA) signals. When the number of components is so large that the central limit theorem is valid, such signals can be approximately considered as Gaussian. It is well known that the above signals have large envelope fluctuations and thus they are widely subject to nonlinear distortions (mainly due to power amplifiers at transmitters).

Thus, whenever a complex-valued signal passes through a nonlinear device, the problem of the analysis of nonlinear distortions arises.

In this paper, we derive theoretical results that provide generalisations of Bussgang's theorem to functions of complex variables. We also give examples of the application of the derived results.

2. GENERALISATION OF BUSSGANG'S THEORY TO FUNCTIONS OF COMPLEX VARIABLES

2.1 Price's theorem extension to the general case of function of many real variables

First we present the Price's theorem generalisation for an arbitrary function of many real variables.

Theorem1: (a Price's theorem extension). We are given the set $\{x_i\}(i=\overline{1,n})$ of jointly normal real variables with means m_i and joint central moments $\rho_{rs}\,(r,s=\overline{1,n})$. If the input-output characteristic of a memoryless device is described by the function $g(x_1,...,x_n)$ and $|g(x_1,...,x_n)|<Ae^{\sum_{i=1}^{n}x_i^{\alpha}}$ (A is a constant and $\alpha<2$), then

$$\frac{\partial^k E(g(x_1,..., x_n))}{\prod_{l=1}^{N}\partial(\rho_{r_l s_l})^{k_l}} = E\left\{\frac{\partial^{2k} g(x_1,..., x_n)}{\prod_{l=1}^{N}(\partial x_{r_l})^{k_l}(\partial x_{s_l})^{k_l}}\right\}$$

$(r_l \neq s_l)$, (1)

where r_l and s_l ($l=\overline{1,N}$) are integers taken from the set $\{1,...,n\}$. The k_l are positive integers and $\sum_{l=1}^{N} k_l = k$, $\delta_{r_l s_l}$ is the Kronecker symbol. The parameter $p = \sum_{l=1}^{N} t_{il} k_l$, t_{il} is the number of times i appears in (r_l, s_l).

The proof directly follows from that given in [3] for a nonlinear function of two real variables.

The formula (1) is convenient to use when the complex representation of the signal is employed.

2.2 An extension of Bussgang's theory to functions of many complex variables

Theorem 2: We consider passing the sum of independent Gaussian signals $z_i(t) = x_i(t) + jy_i(t)$, $(i=\overline{1,n})$ with $\rho_{x_i(t)y_i(t+\tau)} = \rho_{y_i(t)x_i(t+\tau)} = 0$ for $\forall\ t,\tau$ and for $\forall\ i \in \{1,...,n\}$ through the nonlinear memoryless device with the characteristic

$$z_o(t) = F\left(\sum_{i=1}^{n} z_i\right).$$

We introduce the input signal of the nonlinearity

$$z_{inp} = \sum_{i=1}^{n} z_i(t) = x_{inp} + jy_{inp}.$$

If (a) $m_{z_i} = 0$ and at least one of the following conditions holds:

for $\forall\ i$, k ($i,k=\overline{1,n}$):

(b) $E\left(\frac{\partial F}{\partial x_i}\right) = -jE\left(\frac{\partial F}{\partial y_k}\right)$ or/and

(c) $z_i(t)$ are stationary processes and $E\{z_i(t)z_i^*(t+\tau)\} = E\{z_k(t)z_k^*(t+\tau)\}$,

then

$$E\{z_o(t)z_{in}^*(t+\tau)\} = \frac{1}{2n}\sum_{i=1}^{n}(E\left\{\frac{\partial F}{\partial x_i}\right\} - jE\left\{\frac{\partial F}{\partial y_i}\right\})\times$$

$$E\{z_{in}(t)z_{in}^*(t+\tau)\} = \frac{1}{2}\times(E\left\{\frac{\partial F}{\partial x_{inp}}\right\} - jE\left\{\frac{\partial F}{\partial y_{inp}}\right\})\times$$

$$E\{z_{inp}(t)z_{inp}^*(t+\tau)\}. \quad (2)$$

Proof: On the basis of (1) we find that

$$\frac{\partial E\{z_o(t)z_{in}^*(t+\tau)\}}{\partial \rho_{x_i(t)x_i(t+\tau)}} = E\left\{\frac{\partial F}{\partial x_i}\right\},$$

$$\frac{\partial E\{z_o(t)z_{in}^*(t+\tau)\}}{\partial \rho_{y_i(t)y_i(t+\tau)}} = -jE\left\{\frac{\partial F}{\partial y_i}\right\},$$

and the output-input cross correlation

$$E\{z_o(t)z_{in}^*(t+\tau)\} = \sum_{i=1}^{n}(E\left\{\frac{\partial F}{\partial x_i}\right\}\rho_{x_i(t)x_i(t+\tau)} -$$

$$jE\left\{\frac{\partial F}{\partial y_i}\right\}\rho_{y_i(t)y_i(t+\tau)}) + m_{z_i}^* E\{F\}. \quad (3)$$

From (3), under the conditions of the theorem, the validity of (2) directly follows.

By using (2) one can express the output-input cross correlation function via the autocovariances of the components of the input signal.

Formula (2) can be considered as Bussgang's theorem for the nonlinear device driven by the sum of independent complex Gaussian signals.

Corollary: Under the conditions of Theorem 2, the output of the nonlinear device can be represented as:

$$z_o(t) = \alpha z_{inp}(t) + n(t), \quad (4)$$

where $\alpha = \frac{1}{2}E\left\{\frac{\partial F}{\partial x_{inp}} - j\frac{\partial F}{\partial y_{inp}}\right\}$ and

$n(t)$ is nonlinear noise.

Theorem 3: We consider a nonlinear function

$$G(z_1,...,z_n) = F(z_1)L(z_2,...,z_n) \quad , \qquad (5)$$

where L denotes a linear function, and F specifies a nonlinear function,

$z_i(t) = x_i(t) + jy_i(t), \quad (i = \overline{1,n})$ have jointly normal components x_i, y_k and

$\rho_{x_i(t)y_k(t+\tau)} = \rho_{y_i(t)x_k(t+\tau)} = 0$ for $\forall\ t, \tau$ and for $\forall\ i,k \in \{1,...,n\}$.

Then

$$E\{G\} = L\left\{E\left\{\frac{\partial F}{\partial x_1}\right\}\rho_{x_1x_i} + (-1)^{g_i} E\left\{\frac{\partial F}{\partial y_1}\right\}\rho_{y_1y_i} + m_i E\{F\}\right\}, \ i = \overline{2,n}, \qquad (6)$$

where $m_i = m_{zi}$ if the function L depends on z_i and $m_i = m^*_{zi}$ if L depends on z^*_i and the g_i indicates whether L is a function of z_i or its conjugation z^*_i. In the former case $g_i = 0$ otherwise $g_i = 1$.

Proof: From (1) we have

$$G(z_1,...,z_n) = L(z_i F(z_1)) = L\{F(z_1)x_i + (-1)^{g_i} jF(z_1)y_i\} \quad ,$$

$$i = \overline{2,n}, \qquad (7)$$

Then from (1) and (7) the validity of (6) directly follows.

Corollary 1: In a particular case when

$$G(z_1,z_2) = F(z_1)z_2^* \text{ and } z_2 = z_1(t+\tau), \qquad (8)$$

from (7) we find that

$$E\{G\} = E\left\{\frac{\partial F}{\partial x_1}\right\}\rho_{x_1x_2} - jE\left\{\frac{\partial F}{\partial y_1}\right\}\rho_{y_1y_2} + m^*_{z_2}E\{F\}. \qquad (9)$$

Clearly, (9) can be obtained also from (2).

The result agrees with that derived in [6] in another way.

Corollary 2: Let us consider a nonlinear function specified by (9). If additionally to the conditions of Theorem 3 $F(z_1)$ is an analytical function and $m_{z_2}(t) = 0$ for $\forall\ t$ then

$$E\{z_o(t)z_1^*(t+\tau)\} = E\left\{\frac{dF}{dz_1}\right\}E\{z_1(t)z_1^*(t+\tau)\}.$$

Remark 1: It is interesting to note that the condition

$$E\left\{\frac{\partial F}{\partial x_1}\right\} = -jE\left\{\frac{\partial F}{\partial y_1}\right\}$$

holds not only if $F(z_1)$ is an analytical function but also in the case when the device with the input-output characteristic specified by the characteristic

$$F(z_1) = H(|z_1|)e^{j\varphi_1}, \qquad (10)$$

where $H(\cdot)$ is a complex-valued function and $\varphi_1 = \arg(z_1)$, is driven by a stationary Gaussian signal with zero mean [6].

Formula (10) describes an amplitude dependent complex-valued nonlinear characteristic. In communication systems, this is a frequently met in practice case. Such characteristics have, for example, power amplifiers at transmitters that are amplitude dependent nonlinearities.

The validity of condition (b) of Theorem 2 for the characteristic (10) means that Bussgang's theory is applicable to amplitude dependent nonlinear devices fed by complex-valued signals.

Remark 2: If the output-input characteristic of the nonlinear device is specified by (10) and the nonlinearity is driven by the stationary Gaussian signal with zero mean, then it is proven in [6] that the scaling factor α in (4) is defined as

$$\alpha = \frac{1}{2}E\left\{H'(|z_1|) + \frac{H(|z_1|)}{|z_1|}\right\}. \qquad (11)$$

On the other hand, since (2) holds for any τ and thus for $\tau = 0$, we find from (2) that

$$\alpha = \frac{1}{P_{in}}E\{H(|z_1|)\times|z_1|\}, \qquad (12)$$

where $P_{in} = E\{z_1(t)\times z_1^*(t)\}$ is the power of the input signal complex envelope.

It is easy to show (by integrating in parts) that the calculations of the α according to (11) and (12) coincide.

Theorem 4 (Price's theorem extension for a particular case of functions of complex variables):

We consider a nonlinear function

$$G(z_1,...,z_n) = F(z_1)\sum_{i=2}^{n} F_i(z_i),$$

where $F(z_1)$ is an analytical function , $F_i(z_i) = \lambda_i z_i^*$ and λ_i are constants.

If $z_i(t)$ ($i = \overline{1,n}$) are jointly stationary processes, then

$$\frac{\partial E\{G(z_1,...,z_n)\}}{\partial \rho_{z_1 z_i^*}} = \lambda_i E\left\{\frac{dF(z_1)}{dz_1}\right\}.$$

Proof directly follows from Theorem 1.

3. APPLICATION OF THE DERIVED THEORETICAL RESULTS

Bussgang's theorem for the case of a nonlinearity fed by one complex-valued signal has been derived in [6] in another way. Whenever the transfer characteristic of the nonlinear device is known, on the basis of (4) and (11) (or (12)) one can comprehensively analyse the nonlinearity in terms of the output power, the nonlinear noise power and the scaling factor α. On this basis, for example in communication systems, the bit-error rate performance can be derived [6], [7].

Representation of the nonlinearly distorted signal in the form of (4) can find applications for development of signal-processing algorithms for the nonlinear noise shrinkage. Some signal-processing algorithms have been derived on the basis of representation (4) (see, e.g. [8], [9]). In [8], a Kalman filter-based algorithm has been proposed for the mitigation of nonlinear noise. In [9], an effective noise cancellation algorithm has been suggested. The algorithm estimates nonlinear noise and then it is cancelled by the subtraction of the derived estimate from the observation of the nonlinearly distorted signal.

4. CONCLUSIONS

Busgang's theory is a powerful tool for the analysis of Gaussian signals passing through nonlinear devices.

Originally proven for nonlinear functions of real variables, this theory is not extended directly to the case of functions of complex variables.

In this paper, we show that under the certain conditions on the nonlinear characteristic and the input variables, Bussgang's theory is applicable also to functions of many complex variables. Compared with previously derived theoretical results [6], the theorems presented in this paper are more general.

Price's theorem is not also extended directly to functions of complex variables. In this paper, we present an extension of Price's theorem for a particular case of functions of complex variables.

The derived results can be used in various signal-processing systems both for the analysis of nonlinearities as well as for the synthesis of nonlinear noise shrinkage algorithms.

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