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General Formulation for Arbitrary Length Cosine Modulated Filter Banks

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***Abstract*— Cosine modulated filter banks are well known for their low design and implementation cost. In this paper we present a general framework for the theory and design of arbitrary length Cosine Modulated Filter Banks where all the subband filters have linear phase and the prototype filter is of arbitrary length. The necessary and sufficient condition for the Perfect Reconstruction for the arbitrary length CMFB is established.**

I. INTRODUCTION

FILTER banks find applications in number of areas such as subband coding, communication and signal compression and pattern recognition. In a filter bank the input signal is decomposed into a number of adjacent frequency bands using a bank of filters called analysis filter bank. The extracted subband signal is decimated, quantized and coded. This can either be stored or transmitted. A set of filters called synthesis bank is used to reconstruct the original signal. The decimation introduces aliasing in the subbands and it has been shown that perfect reconstruction is possible even in the presence of aliasing.

Digital Filter banks have been extensively studied [1][2]. The closed form solutions for conventional M-channel Filter banks with perfect reconstruction are well known [1]. They usually lead to very complex design and implementation.

Recently, the perfect reconstruction cosine modulated filter banks (CMFB) have emerged as an attractive choice for filter banks because of its inherent implementation and design ease [1][2][3]. In this system, the analysis filters, $h_k(n)$, and synthesis filters, $f_k(n)$, are the cosine modulated versions of a single prototype filter. In [4][5] it has been shown that an M-Channel CMFB with FIR analysis and synthesis filters can give PR as long as prototype filter length is 2M. Later, in [6] this is extended to 2mM, where m is an arbitrary positive integer. In [2] a CMFB is proposed having a filter length $N=2mM+m_1$, where m_1 is an arbitrary integer and $1\leq m_1\leq 2M-1$. In this system, the subband filters do not have linear phase. Linear phase property of individual filters is preferred for image coding applications. In [7] a CMFB structure with linear phase sub filters has been proposed. But the length of the filters is restricted to 2mM+M. In this paper we establish the perfect reconstruction criteria for an arbitrary length linear phase Cosine Modulated Filter Bank.

This paper is organized as follows: In section II, classical CMFB is reviewed. Section III, covers the general theory of proposed arbitrary length CMFB. Section IV shows the polyphase representation of entire filter bank and PR condition is derived. Section V presents the prototype filter design criteria and Section VI concludes the paper.

II. PSEUDO QMF COSINE MODULATED FILTER BANKS

In this section, we briefly describe the cosine modulated filter banks used in the pseudo QMF Filter banks. The analysis filters, $H_k(z)$, and the synthesis filters, $F_k(z)$, of the filter bank are obtained by the cosine modulation of a linear phase low pass prototype filter H(z) as follows

$$h_k(n) = 2h(n)\cos\left((2k+1)\frac{\pi}{2M}\left(n-\frac{N-1}{2}\right)+(-1)^k\frac{\pi}{4}\right)$$

$$f_k(n) = 2h(n)\cos\left((2k+1)\frac{\pi}{2M}\left(n-\frac{N-1}{2}\right)-(-1)^k\frac{\pi}{4}\right)$$

$$0\leq k\leq M-1 \qquad 0\leq n\leq N-1$$

where $h_k(n)$ and $f_k(n)$ are the impulse response of $H_k(z)$ and $F_k(z)$.

This is an approximate reconstruction system. The individual filters do not have linear phase and the aliasing is not completely eliminated.

A. Linear Phase CMFB

Linear phase CMFB was proposed by Lin [7] and is shown in the Fig.1. The new filter bank structure can be considered as a connection of two subsystems in which first subsystem has M+1 channels and the second has M-1 channels or vice-versa. Individual filters have linear phase and are the modulated versions of a prototype filter. The synthesis filters are the time-reversed versions of analysis filters. The spectral support of the entire CMFB is shown in Fig.2. This system gives perfect reconstruction only when the length of the prototype filter is restricted to N=2mM+M. In the next section we extend this structure for the filter length $N=2mM+m_1$, where $1\leq m_1\leq 2M-1$, while preserving the linear phase property of the individual filters.

III. Arbitrary Length Liner Phase CMFB

The filter bank structure we used is shown in Fig.1. The analysis and the synthesis filters are derived from a linear phase prototype filter h(n) using cosine and sine modulation.

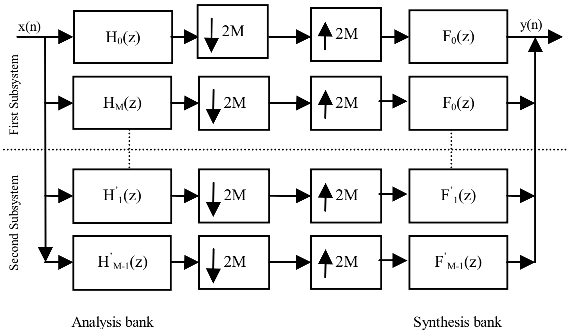

Fig.1. 2M-Channel Filter bank structure with individual filters having linear phase.

The prototype filter h(n) is modulated as follows;

$$h_k(n) = 2c_k h(n)\cos\left(\frac{\pi k}{M}\left(n - \frac{N+M-1}{2}\right)\right)$$

$$0 \le k \le R' \qquad (1)$$

$$h'_k(n) = 2c_k h(n-M)\sin\left(\frac{\pi k}{M}\left(n - M - \frac{N+M-1}{2}\right)\right)$$

$$1 \le k \le R'' \qquad (2)$$

where $c_k = \begin{cases} 1 & k \ne 0, M \\ 1/\sqrt{2} & k = 0, M \end{cases}$

$$R' = \begin{cases} M & N+M-1 \text{ is even} \\ M-1 & N+M-1 \text{ is odd} \end{cases}$$

$$R'' = \begin{cases} M-1 & N+M-1 \text{ is even} \\ M & N+M-1 \text{ is odd} \end{cases} \qquad (3)$$

The synthesis filters, $F_k(z)$, are defined as

$$f_k(n) = h_k(N+M-1-n) \quad 0 \le k \le R' \qquad (4)$$

$$f'_k(n) = -h'_k(N+M-1-n) \quad 1 \le k \le R'' \qquad (5)$$

IV. Polyphase Representation

The Polyphase representation of the prototype filter H(z) is

$$H(z) = \sum_{n=0}^{2M-1} G_n\left(z^{2M}\right) z^{-n} \qquad (6)$$

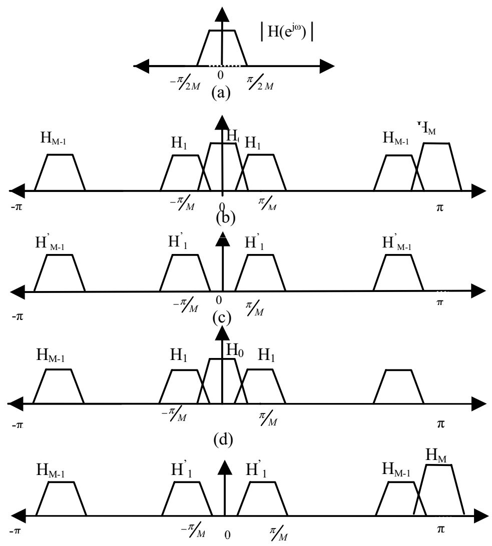

Fig. 2. Magnitude Response of the low pass filter and 2M-channel linear phase CMFB. (a) The low-pass prototype filter.(b) and (c) The analysis filters when N+M is odd. (d) and (e) The analysis filters when N+M is even.

where $G_n\left(z^{2M}\right)$ is the n^{th} type 1 polyphase component of *H(z)*.

Analysis filter bank of the first subsystem can be written as follows

$$H_k(z) = 2\sum_{n=0}^{2M-1} G_n\left(z^{2M}\right) z^{-n} \cos\left[\frac{\pi k}{M}(n-x)\right]$$

$$\text{where } k = \begin{cases} 0,1\ldots M & N+M-1 \text{ is even} \\ 0,1\ldots M-1 & N+M-1 \text{ is odd} \end{cases} \qquad (7)$$

Analysis filter bank of the second subsystem can be represented as

$$H'_k(z) = 2z^{-M}\sum_{n=0}^{2M-1} G_n\left(z^{2M}\right) z^{-n} \sin\left[\frac{\pi k}{M}(n-x)\right] \qquad (8)$$

$$\text{where } k = \begin{cases} 1\ldots M-1 & N+M-1 \text{ is even} \\ 1\ldots M & N+M-1 \text{ is odd} \end{cases}$$

$$x = \left(\frac{N+M-1}{2}\right) \qquad (9)$$

The entire analysis bank can be expressed as follows

$$h(z) = [H_0(z), \ldots H_M(z), H'_1(z) \ldots H'_{M-1}(z)]^T \qquad (10)$$

The polyphase structure of entire analysis bank is to be determined for two cases as shown below.

Case 1: *(M+N-1) is even*

The analysis filter bank h(z) can be written as

$$h(z)=2\begin{pmatrix} C & \Lambda_1 C \\ z^{-M}S & \Lambda_2 z^{-M}S \end{pmatrix}\begin{pmatrix} g_0(z^{2M}) & 0 \\ 0 & g_1(z^{2M}) \end{pmatrix}\begin{pmatrix} e(z) \\ z^{-M}e(z) \end{pmatrix} \tag{11}$$

where

$$C_{k,n}=\cos\left(\frac{\pi k(n-x)}{M}\right),\quad [\Lambda_1]_{kk}=(-1)^k,\ 0\le k\le R'$$

$$S_{k,n}=\sin\left(\frac{\pi k(n-x)}{M}\right),\ [\Lambda_2]_{kk}=(-1)^k,\ 1\le k\le R''$$

$$[g_0(z)]_{kk}=G_k(z),\ [g_1(z)]_{kk}=G_{k+M}(z),\ \mathrm{k}=0,1\ldots,M-1 \tag{12}$$

Then the type I Polyphase matrix of the analysis bank is

$$E(z)=\begin{pmatrix} Cg_0(z) & \Lambda_1 Cg_1(z) \\ z^{-1}\Lambda_2 Sg_1(z) & Sg_0(z) \end{pmatrix} \tag{13}$$

Since the synthesis filters are the time-reversed versions of the analysis filters, type-II polyphase form the synthesis filter bank is expressed as

$$R(z)=z^{-(N+M)}\widetilde{E}(z) \tag{14}$$

where the symbol ~ represents transpose conjugation. For a perfect reconstruction filter bank $R(z)E(z)=c\mathrm{I}$ for some value of *c*.

The necessary condition for Perfect Reconstruction of this case can be derived as shown below

$$z^{(M+N)}R(z)\widetilde{E}(z)=$$

$$\begin{pmatrix} \widetilde{g}_0(z)C^TCg_0(z)+\widetilde{g}_1(z)S^TSg_1(z) & \widetilde{g}_0(z)C^T\Lambda_1Cg_1(z)+z\widetilde{g}_1(z)S^T\Lambda_2Sg_0(z) \\ \widetilde{g}_1(z)C^T\Lambda_1Cg_0(z)+z^{-1}\widetilde{g}_0(z)S^T\Lambda_2Sg_1(z) & \widetilde{g}_1(z)C^TCg_1(z)+\widetilde{g}_0(z)S^TSg_0(z) \end{pmatrix} \tag{15}$$

The equation (15) is solved for both cases when $0\le m_1<M$ and $M\le m_1<2M$

For $0\le m_1<M$

$$S^TS=2M\left(\mathrm{I_M}-\begin{bmatrix}0 & 0\\ 0 & \mathrm{J_{M\text{-}m1}}\end{bmatrix}\right)\quad C^T\Lambda_1C=2M\left(\begin{bmatrix}\mathrm{J_{m1}} & 0\\ 0 & 0\end{bmatrix}\right)_{\mathrm{M\times M}}$$

$$C^TC=2M\left(\mathrm{I_M}+\begin{bmatrix}0 & 0\\ 0 & \mathrm{J_{M\text{-}m1}}\end{bmatrix}\right)\quad S^T\Lambda_1S=2M\left(\begin{bmatrix}\text{-}\mathrm{J_{m1}} & 0\\ 0 & 0\end{bmatrix}\right)_{\mathrm{M\times M}} \tag{16}$$

For $M\le m_1<2M$

$$S^TS=2M\left(\mathrm{I_M}-\begin{bmatrix}\mathrm{J_{m1-M}} & 0\\ 0 & 0\end{bmatrix}\right)\quad C^T\Lambda_1C=2M\left(\begin{bmatrix}0 & 0\\ 0 & \mathrm{J_{2M\text{-}m1}}\end{bmatrix}\right)_{\mathrm{M\times M}}$$

$$C^TC=2M\left(\mathrm{I_M}+\begin{bmatrix}\mathrm{J_{m1-M}} & 0\\ 0 & 0\end{bmatrix}\right)\quad S^T\Lambda_1S=2M\left(\begin{bmatrix}0 & 0\\ 0 & \text{-}\mathrm{J_{2M\text{-}m1}}\end{bmatrix}\right)_{\mathrm{M\times M}} \tag{17}$$

Using the above results (16) and (17), the cross diagonal elements of (15) can be expressed as below

For $0\le m_1<M$

$$\widetilde{g}_0(z)C^T\Lambda_1Cg_1(z)+z\widetilde{g}_1(z)S^T\Lambda_2Sg_0(z)=$$

$$2Mz\left\{\widetilde{g}_1(z)\begin{bmatrix}\text{-}\mathrm{J_{m1}} & 0\\ 0 & 0\end{bmatrix}g_0(z)+z^{-1}\widetilde{g}_0(z)\begin{bmatrix}\mathrm{J_{m1}} & 0\\ 0 & 0\end{bmatrix}g_1(z)\right\} \tag{18}$$

$$\widetilde{g}_1(z)C^T\Lambda_1Cg_0(z)+z^{-1}\widetilde{g}_0(z)S^T\Lambda_2Sg_1(z)=$$

$$2M\left\{\widetilde{g}_1(z)\begin{bmatrix}\mathrm{J_{m1}} & 0\\ 0 & 0\end{bmatrix}g_0(z)+z^{-1}\widetilde{g}_0(z)\begin{bmatrix}\text{-}\mathrm{J_{m1}} & 0\\ 0 & 0\end{bmatrix}g_1(z)\right\} \tag{19}$$

Since the prototype filter is symmetric, the polyphase components of prototype filter, $G_k(z)$, exhibit the following relationship.

$$\widetilde{G}_k(z)=\begin{cases} z^mG_{m_2-1-k}(z), & 0\le k\le m_1-1 \\ z^{m-1}G_{2M+m_1-1-k}(z), & m_1\le k\le 2M-1 \end{cases} \tag{20}$$

By using the above property of $G_k(z)$ we can prove that both (18) and (19) are zero matrices

Evaluating the diagonal elements of (15)

$$\widetilde{g}_0(z)C^TCg_0(z)+\widetilde{g}_1(z)S^TSg_1(z)=$$

$$2M\left\{\begin{matrix}\widetilde{g}_0(z)\left(\left(\mathrm{I_M}+\begin{bmatrix}0 & 0\\ 0 & \mathrm{J_{M\text{-}m1}}\end{bmatrix}\right)\right)g_0(z)+\\ \widetilde{g}_1(z)\left(\mathrm{I_M}-\begin{bmatrix}0 & 0\\ 0 & \mathrm{J_{M\text{-}m1}}\end{bmatrix}\right)g_1(z)\end{matrix}\right\} \tag{21}$$

$$\widetilde{g}_0(z)C^TCg_0(z)+\widetilde{g}_1(z)S^TSg_1(z)=$$

$$2M\left\{\begin{matrix}\widetilde{g}_1(z)\left(\left(\mathrm{I_M}+\begin{bmatrix}0 & 0\\ 0 & \mathrm{J_{M\text{-}m1}}\end{bmatrix}\right)\right)g_1(z)+\\ \widetilde{g}_0(z)\left(\mathrm{I_M}-\begin{bmatrix}0 & 0\\ 0 & \mathrm{J_{M\text{-}m1}}\end{bmatrix}\right)g_0(z)\end{matrix}\right\} \tag{22}$$

Using the relation shown in (20), we can prove that the necessary condition to make (15) equal to $\mathrm{I_{2M}}$ is

$$\tilde{G}_k(z)G_k(z) + \tilde{G}_{k+M}(z)G_{k+M}(z) = \frac{1}{2M}$$
$$k = 0,1,2.....M-1 \qquad (23)$$

Similarly, for case when $M \le m_1 < 2M$ the off-diagonal elements can be proved to be zero and the necessary and the sufficient condition to make (15) equal to I_{2M} is the same as shown in (23).

Case 2: *(M+N-1) is odd*

The analysis filter bank h(z) can be written as

$$h(z) = 2\begin{pmatrix} C & \Lambda_1 C \\ z^{-M}S & \Lambda_2 z^{-M} S \end{pmatrix}\begin{pmatrix} g_0(z^{2M}) & 0 \\ 0 & g_1(z^{2M}) \end{pmatrix}\begin{pmatrix} e(z) \\ z^{-M}e(z) \end{pmatrix} \qquad (24)$$

where $g_0(z)$ and $g_1(z)$, Λ_1 and Λ_2, C and S are defined in (12) for the different values of R' and R''. Similarly the necessary and the sufficient condition to obtain the perfect reconstruction for this case can be derived by solving (15), using (18), (19) and (20) for both the cases, $0 \le m_1 < M$ and $M \le m_1 < 2M$. The necessary and the sufficient condition for PR in this case also found to be the same as that shown in (23)

V. Prototype Filter Design

In a filter bank perfect reconstruction is possible if and only if the polyphase components of the prototype filter satisfy the power complementary condition (23). Apart from the perfect reconstruction the performance of the filter bank is also characterized by the stopband attenuation of the sub filters and is related to the stopband attenuation of the prototype filter. A lowpass prototype filter satisfying the PR condition can be obtained by Quadratic Constraint Least Square (QLS) method [8]. The prototype filter design problem can be formulated as a least square optimization with quadratic constraints as shown below.

$$\min\left(\frac{1}{\pi - \omega_s}\int_{\omega_s}^{\pi}\left|H(e^{j\omega})\right|^2 d\omega\right) \text{ subject to (23)} \qquad (25)$$

where ω_s is the stopband frequency of prototype filter.

Example:- The frequency response of a the linear phase prototype filter satisfying the PR condition (23), obtained by solving (24) for M=4, m=5, m_1 =2 is shown in Fig.3.

VI. Conclusion

We have presented a general frame work for the theory of arbitrary length cosine modulated filter bank having individual filters with linear phase. The PR condition for conventional CMFB having filter length 2mM has

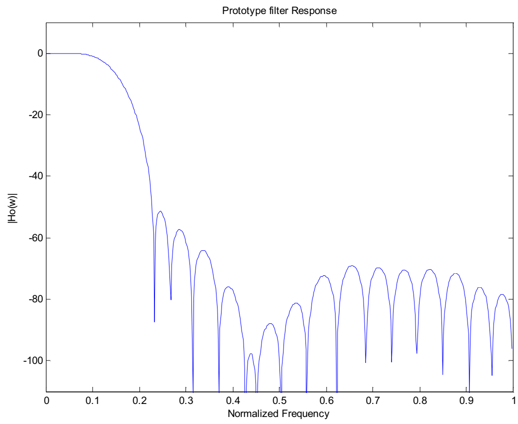

Fig.3. Frequency Response of the prototype filter in the example

already been derived. In this paper, the necessary and sufficient condition for perfect reconstruction of an arbitrary length ($2mM+m_1$) cosine modulated filter bank is established, while keeping the linear phase property of individual filters. The PR condition for even and odd parity combination of M and N is proved to be same.

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