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MEASURING SHIFTABILITY OF FRAMES OF REGULAR TRANSLATES

J. Sampo[1], *J.-K. Kamarainen*[2], *M. Heiliö*[1] *and H. Kälviäinen*[2]

[1] Laboratory of Applied Mathematics, [2] Laboratory of Information Processing,
Department of Information Technology,
Lappeenranta University of Technology,
P.O. Box 20, FIN-53851 Lappeenranta, FINLAND,
{sampo,jkamarai,heilio,kalviai}@lut.fi

ABSTRACT

In recent studies, the orthogonality property of certain bases, such as wavelets, has been claimed to produce insuperable disadvantages to some signal processing tasks, e.g., feature extraction. To overcome these disadvantages research interest has turned towards redundant bases, frames, and shiftability concept has been proposed as the preferred property to be achieved by relaxing the orthogonality requirement. The presented results in this work continue the preliminary studies on shiftability. Previous definition is extended to frames of arbitrary regular translates and shiftability measures are proposed for such frames. The shiftability is also considered for finite dimensional discrete frames for which numerical shiftability values are computed.

1. INTRODUCTION

Wavelet transforms and multiresolution analysis (MRA) have become very popular in signal processing. Transforms are typically based on orthogonal function families, such as wavelet bases [5]. However, there are some application areas where the orthogonality requirement produces a problem, and where non-orthogonal function families, such as wavelet frames [4], have been found more useful. This is the case for example in image processing and especially in feature extraction where redundant Gabor frames have been successfully utilized for particularly long time (e.g. [6, 8, 10]).

Shiftability is a property which cannot be achieved with orthogonal compactly supported functions but which is actually a consequence from increasing redundancy. The shiftability concept was first introduced by Simoncelli et al. [9] in a connection to steerable filters [7]. If a shift operation is applied to a signal in a shiftable space then its coefficients energy should redistribute within each sub-band and not across the entire space. For example, a translation of a signal should not redistribute coefficients over scales; energy should remain within each scale. For wavelets a similar property has been sometimes referenced to as translation invariance [2], but generally shiftability may cover operations other than only translation, e.g., rotation and scale in two dimensions. It is evident that proper shiftability measures must be defined in order to select the most suitable frame basis function families for invariant feature extraction systems, e.g., the one proposed by authors in [8].

It seems that the exact shiftability over several different shift operations, i.e., joint-shiftability, cannot be achieved in the exact form, but it can be only approximated. At first, computational shiftability measures must be defined in order to define an approximate shiftability. First results towards practical shiftability measures and approximate shiftability have been made for frames of regular integer translates in [2, 11, 12]. This study continues these results by proposing a more practical lower bound measure and further generalizes measures to regular frames of arbitrary translates. In the experiments several function families are compared based on the proposed measures.

2. SHIFTABILITY

2.1 Shiftable frames

Frame is basically a generalization of unconditional bases, that is, functions which span the space do not have to be linearly independent. A strict definition can be written as follows.

Definition 1 *Let Λ be a (countable) index set and $\mathscr{H}$ Hilbert space. A set of functions $\{\phi_i \mid i \in \Lambda\}$ is called as a frame for $\mathscr{H}$ if $\exists A, B \in \mathbb{R}_+$ such that $\forall f \in \mathscr{H}$*

$$A\|f\|^2 \leq \sum_{i\in\Lambda} \left|\langle f, \phi_i \rangle\right|^2 \leq B\|f\|^2 \quad . \tag{1}$$

A corollary from the definition of frame is that if $\{\phi_i \mid i \in \Lambda\}$ is a frame for $\mathscr{H}$ then

$$\mathscr{H} = \overline{\mathrm{span}\{\phi_i \mid i \in \Lambda\}} \quad . \tag{2}$$

Furthermore, if $H \supseteq \mathscr{H}$ is Hilbert space with the same inner product as $\mathscr{H}$, then an orthogonal projection $\mathscr{P}_{\mathscr{H}} : H \to \mathscr{H}$ is

$$\mathscr{P}_{\mathscr{H}} f = \sum_{i\in\Lambda} \langle f, \phi_i \rangle S^{-1}\phi_i = \sum_{i\in\Lambda} \langle f, S^{-1}\phi_i \rangle \phi_i \tag{3}$$

where S^{-1} is the inverse of a frame operator S. Moreover, if $\{\phi_i \mid i \in \Lambda\}$ is a frame for $\mathscr{H}$ with the frame operator S, then $\{S^{-1}\phi_i \mid i \in \Lambda\}$ is also a frame for $\mathscr{H}$. More details about the general frames can be found in [3].

In the following discussion it is assumed that the Hilbert space H is a space of functions $f : \mathbb{R}^d \to \mathbb{C}$. In addition, a translation operator $\tau_s : H \to H$ is defined as

$$\tau_s f(x) := f(x+s), \quad \forall s \in \mathbb{R}^d \tag{4}$$

and the frame functions ϕ_i are

$$\phi_i := \tau_{bi}\phi, \quad \forall i \in \Lambda = \mathbb{Z}^d \tag{5}$$

where $\phi \in H$ is a fixed function with a translation step length $b \in \mathbb{R}_+$. The frame of form

$$\{\phi_i \mid i \in \Lambda\} = \{\tau_{bi}\phi \mid i \in \mathrm{Z}^d\} \tag{6}$$

is called a frame of regular translates and ϕ is its generating function. These kind of frames are very important in practice since the frame $\{\phi_i \mid i \in \Lambda\}$ can be considered as a set of filters in evenly distributed locations, e.g., Gabor filters in [8]. Now, a shiftable frame of regular translates can be defined.

Definition 2 *Let $\{\phi_i \mid i \in \mathbb{Z}^d\}$ be a frame for $\mathscr{H}$. If $\tau_s f \in \mathscr{H}$ for all $f \in \mathscr{H}$ and for all $s \in \mathbb{R}^d$, then $\{\phi_i \mid i \in \mathbb{Z}^d\}$ is a shiftable frame.*

A similar definition holds for frames of irregular translates or even for the general frames. An equivalent condition for Definition 2 can be stated directly via generating function ϕ, which is

$$\mathscr{P}_{\mathscr{H}}\tau_s\phi = \sum_{i\in\mathbb{Z}^d}\langle\tau_s\phi, S^{-1}\phi_i\rangle\tau_{bi}\phi = \sum_{i\in\mathbb{Z}^d} c_{i,s}\phi_i = \tau_s\phi \quad , \tag{7}$$

that is, an arbitrary shift of ϕ can be represented by the same function in regularly distributed points.

2.2 Approximate shiftability

It should be noted that the following considers regular frames as defined in (6).

ϕ is supposed to be normalized, that is $\|\phi\| = 1$. If the exact reconstruction in (7) is not satisfied but there exists an error term ε_s as

$$\tau_s\phi = \mathscr{P}_{\mathscr{H}}\tau_s\phi + \varepsilon_s = \sum_{i\in\mathbb{Z}^d} c_{i,s}\phi_i + \varepsilon_s \tag{8}$$

where $\|\varepsilon_s\| = \sqrt{|\langle\varepsilon_s,\varepsilon_s\rangle|} > 0$ is relatively small, then ϕ is approximately shiftable and respectively $\{\tau_{bi}\phi \mid i \in \mathbb{Z}^d\}$ is an approximately shiftable frame (of regular translates).

Now, $\|\varepsilon_s\|$ is the distance between $\tau_s\phi$ and $\mathscr{H}$, and thus, it is natural to define degree of shiftability of ϕ using $\|\varepsilon_s\|$. For approximate shiftability there are at least two meaningful ways to define the degree of shiftability of ϕ. The first one is a worst case measure

$$r_{\mathscr{H},\infty}(\phi) := \inf_{s\in\mathbb{R}^d}\|\mathscr{P}_{\mathscr{H}}\tau_s\phi\|^2 \tag{9}$$

and the second one an average measure

$$r_{\mathscr{H},1}(\phi) := \lim_{n\to\infty}\frac{1}{m(\Gamma_n)}\int_{\Gamma_n}\|\mathscr{P}_{\mathscr{H}}\tau_s\phi\|^2 ds \tag{10}$$

where integral is taken respect to the measure m, $\Gamma_n \subset \Gamma_{n+1}$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty}\Gamma_n = \mathbb{R}^d$.

Now, if $r_{\mathscr{H},\infty}(\phi) = r_{\mathscr{H},1}(\phi) = 1$ then ϕ is shiftable. If $r_{\mathscr{H},1}(\phi) \approx 1$ then ϕ is considered as approximately shiftable with the shiftability value $r_{\mathscr{H},1}(\phi)$. If $r_{\mathscr{H},\infty}(\phi) \approx 1$ we say that ϕ is approximately shiftable with the lower bound $r_{\mathscr{H},\infty}(\phi)$.

Furthermore, $\|\mathscr{P}_{\mathscr{H}}\tau_s\phi\| = \|\mathscr{P}_{\mathscr{H}}\tau_{s+k}\phi\|$ for all $k \in \mathbb{Z}^d$, and thus, the shiftability measures can be simplified to forms

$$r_{\mathscr{H},\infty}(\phi) = \inf_{s\in[-b/2,b/2]^d}\|\mathscr{P}_{\mathscr{H}}\tau_s\phi\|^2 \quad , \tag{11}$$

$$r_{\mathscr{H},1}(\phi) = \frac{1}{b^d}\int_{[-b/2,b/2]^d}\|\mathscr{P}_{\mathscr{H}}\tau_s\phi\|^2 ds \quad . \tag{12}$$

Wang's shiftability measure r in [12] can be formed as a special case of (12) by setting $b = d = 1$ and $\{\phi_i \mid i \in Z\}$ forming a basis.

2.3 Frames of regular translates in $L_2(\mathbb{R}^d)$

In this section the frames of regular translates for $\mathscr{H} \subset L_2(\mathbb{R}^d)$ are examined. The inner-product $\langle\cdot,\cdot\rangle : L_2(\mathbb{R}^d) \to \mathbb{R}$ is defined as

$$\langle f,g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx \tag{13}$$

where the integral is taken with respect to Lebesgue measure.

First, it is clear that

$$\langle\tau_s f,g\rangle = \langle f,\tau_{-s}g\rangle \quad \forall f,g \in L_2(\mathbb{R}^d),\ \forall s \in \mathbb{R}^d \tag{14}$$

and

$$\{s' + bm \mid s' \in [-b/2,b/2]^d,\ m \in \mathbb{Z}^d\} = \mathbb{R}^d \tag{15}$$

for all $b \in \mathbb{R}_+$.

2.3.1 Signal detection

To investigate if a function $g \in \mathscr{H}$ can be $\tau_s f$ for a fixed $f \in \mathscr{H}$ with an arbitrary $s = s' + bm$, it is sufficient to investigate inner products $\langle\tau_s f - g, \phi_k\rangle$ since this is zero if $\tau_s f = g$. By using (14), (15), (5) and (8) we get that a necessary condition for g being $\tau_s f$ is

$$\sum_{i\in\mathbb{Z}^d} c_{i,-s'}\langle f,\phi_{i+k-m}\rangle + \langle f,\tau_{(k-m)b}\varepsilon_{-s'}\rangle = \langle g,\phi_k\rangle \quad . \tag{16}$$

Since $k \in \mathbb{Z}^d$ is arbitrary, (16) should hold for all $k \in \mathbb{Z}^d$.

It should be noted that the coefficients $c_{i,-s'}$ are independent of g and f and can be thus calculated in advance and that

$$|\langle f,\tau_{(k-m)b}\varepsilon_{-s'}\rangle| \le \|f\|\cdot\|\varepsilon\| \quad . \tag{17}$$

This result means that if a good approximation $\tilde{s}$ for s can be found, such that coefficient $c_{i,-\tilde{s}'}$ do not differ too much from $c_{i,-s'}$, then the computational work can be reduced by removing the term $\langle f,\tau_{(k-m)b}\varepsilon_{-s'}\rangle$ from (16) if ϕ is approximately shiftable.

2.3.2 Multiresolution analysis

Frame multiresolution analysis divides $L_2(\mathbb{R}^d)$ into a collection of sub-spaces $\mathscr{W}_j$. Each sub-space $\mathscr{W}_j$ is generated by a function ψ_j which is a special type of linear combination of scaled and translated versions of ϕ. Some details of multiresolution analysis can be found from [1] and here only some properties which follow from the shiftability or approximate shiftability of ϕ are considered.

If ϕ is approximately shiftable it is obvious that also functions ψ_j are approximately shiftable. A straight consequence is that $\|\mathscr{P}_{\mathscr{W}_j}\tau_s f\| \approx \|\mathscr{P}_{\mathscr{W}_j} f\|$ for all $f \in L_2(\mathbb{R}^d)$, and it can be checked if an unknown g can be $\tau_s f$ for some s (f is fixed) by comparing the norms $\|\mathscr{P}_{\mathscr{W}_j} f\|$ and $\|\mathscr{P}_{\mathscr{W}_j} g\|$. Of course $\|\mathscr{P}_{\mathscr{W}_j} f\| = \|\mathscr{P}_{\mathscr{W}_j} g\|$ does not guarantee that $\tau_s f = g$, but the result can still be useful since it is usually hard to find a good approximation for s'.

2.4 Finite dimensional frames

In practical applications an infinite number of ϕ_i cannot be realized. In that case the shiftability measures cannot be simplified as was shown in the previous sections but the assumptions must be revised. First, a sequence $\{\mathscr{H}_n\}_{n=0}^{\infty}$ of nested subspaces is defined in $\mathscr{H}$ by

$$\mathscr{H}_n := \text{span}\{\phi_i \mid i \in \{0,\dots,n\}^d\}, \quad \forall n \in \mathbb{N} \qquad (18)$$

and the center of the hyper-cube $I_n = [0, bn]^d$ is denoted by c_n. It is easy to see that

$$\inf_{s \in c_{n/2}+I_{n/2}} \|\mathscr{P}_{\mathscr{H}_n} \tau_s \phi\|^2 \geq \inf_{s \in c_{n/2}+[-b/2,b/2]^d} \|\mathscr{P}_{\mathscr{H}_{(n-n')/2}} \tau_s \phi\|^2 \qquad (19)$$

where $n' = 0$ for even and $n' = -1$ for odd n. The inequality in (19) provides a tool to estimate how good approximation in $\mathscr{H}_n$ can be achieved for a function $\tau_{s'} f$ if $f \in \mathscr{H}_n$ is consentrated in the middle of $\cup_{i=0}^{n} supp(\phi_i)$. The estimation is not necessarily accurate but it motivates to measure shiftability of $\mathscr{H}_n$ by a functional

$$\tilde{r}_{\mathscr{H}_n,\infty}(\phi) := \inf_{s \in c_{(n-n')/2}+[-b/2,b/2]^d} \|\mathscr{P}_{\mathscr{H}_{(n-n')/2}} \tau_s \phi\|^2 \ . \qquad (20)$$

Note that it would take $\mathscr{O}(n^d)$ more time to calculate $\inf_{s \in [0,bn]^d} \|\mathscr{P}_{\mathscr{H}_n} \tau_s \phi\|^2$ than $\tilde{r}_{\mathscr{H}_n,\infty}(\phi)$. Also

$$\lim_{n \to \infty} \tilde{r}_{\mathscr{H}_n,\infty}(\phi) = r_{\mathscr{H},\infty}(\phi) \qquad (21)$$

and convergence rate of (21) tells how large n should be selected for a good approximation of $r_{\mathscr{H},\infty}(\phi)$. Similarly,

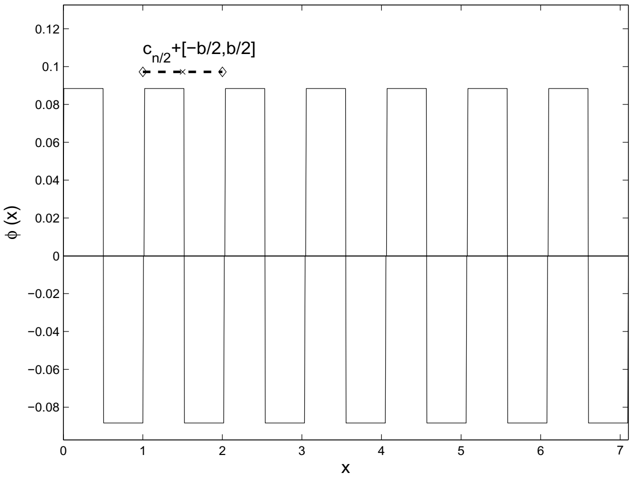
$$\tilde{r}_{\mathscr{H}_n,1}(\phi) := \frac{1}{b^d} \int_{c_{(n-n')/2}+[-b/2,b/2]^d} \|\mathscr{P}_{\mathscr{H}_{(n-n')/2}} \tau_s \phi\|^2 ds \ . \qquad (22)$$

3. EXAMPLES

In this section, the lower bound and average shiftability measures in (20) and (22) are demonstrated for finite frames of integer-translates. Shiftability behavior was examined as a function of the number of frame basis functions n to study how the length of a frame affects shiftability (signal detection accuracy in the middle of frame).

In Fig. 1 it is shown an example of frame basis functions ($n = 7$). The frame basis functions are Haar functions and as integer-translates they constitute also a basis. The functions also constitute a finite regular frame (of integer-translates) and the shiftability integration region from (20) is marked in Fig. 1.

Shiftability of regular frames constructed from different basis functions was inspected for different number of frame basis functions. In Fig. 2(a) there are shown average shiftability values in (22) and in Fig. 2(b) lower bound shiftability values in (20) for the three different type of frames (Haar, Meyer, biorthogonal spline). There were significant differences between average and lower bound shiftability values for all types of (orthogonal) frames as shown in Figs. 2(a) and 2(b), and thus, a system based on the average shiftability may have a poor accuracy if some certain shifts occur.



(a)

Figure 1: $n = 7$ Haar functions constituting a frame of regular translates ($b = 1$).

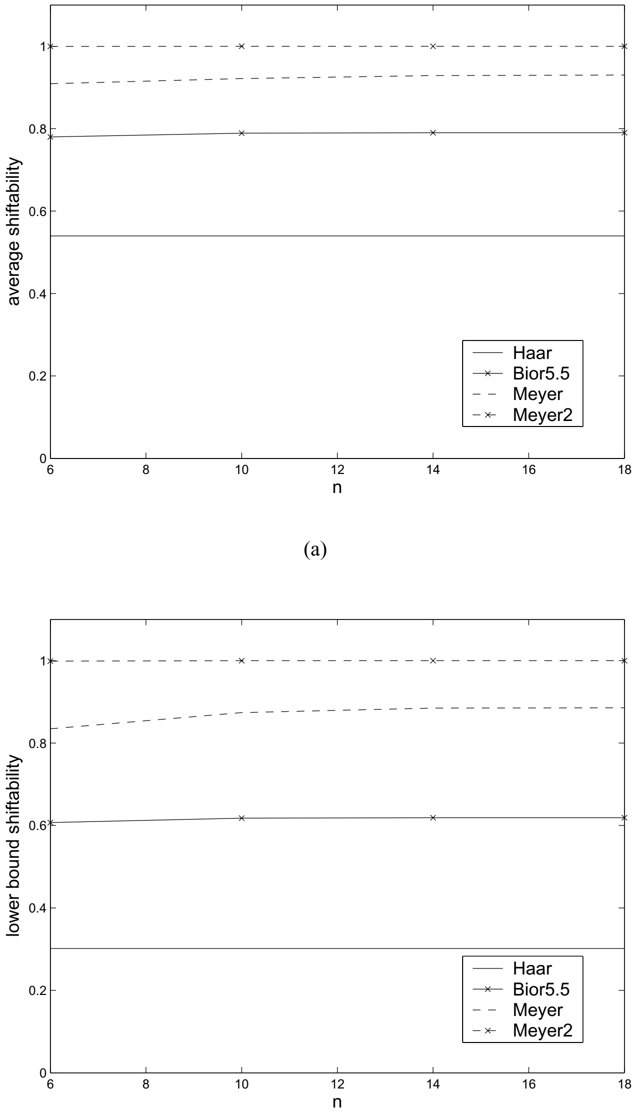
Furthermore, the increase of the number of frame basis functions had none or very small effect to the shiftability. This can be explained by the short compact support and fast decay of the selected wavelet scaling functions; shiftability cannot be improved with new functions if they do not overlap with the shiftability integration region. For example, Haar has a compact support of length 1, and thus, a frame based on integer-translates has a constant shiftability for all values of n (Figs. 2(a)-2(b)). For frame basis functions with a particularly long or even infinite support, such as Meyer wavelet, some improvement can be introduced by increasing the number of functions, but typically this has no significant effect due to the rapid decay of functions themselves.

For these functions a significant improvement of the shiftability can be achieved only by relaxing the orthogonality property, for example, by spacing frame basis functions in a more dense manner where they have a significant overlap. This is demonstrated in Fig. 2 for Meyer wavelet functions where a computational shiftability ($\tilde{r}_{\mathscr{H}_n,\infty}(\phi) \approx \tilde{r}_{\mathscr{H}_n,1}(\phi) \approx 1$) was achieved (*Meyer*2 in Fig. 2 where $b = \frac{1}{4}$).

4. CONCLUSIONS AND DISCUSSION

In this study the shiftability concept was examined. More precisely, the shiftability was inspected for frames of regular translates. The study also presents functionals for measuring shiftability of these frames. In addition, the measures were also formulated for finite dimensions and numerical results were shown for several different type of frames. It is evident that for the demonstrated basis functions a significant improvement to shiftability can be achieved only by relaxing the orthogonality property. Also when dimension of finite dimensional frame grows the shiftablity measures seems to converge very fast.

This study provides only a brief treatment to such a concept and the current motivation is to extend the present theory to meet requirements of systems based on frames. For example, in feature extraction shiftability measures may provide a more quantitative rationale where the selection of feature extraction frame structures can be based on to capture events, such as objects in digital images; if a proper shiftability measure can be established to measure the joint-shiftability of translation, rotation, and scaling of a given feature space, e.g., Gabor filters, it may be possible to select optimal feature parameters and tune the feature extraction system to a desired reliability and accuracy. A derivation of such joint-shiftability measure for discrete frames of integer and arbitrary translation basis functions will be addressed in future work.



(a)

(b)

Figure 2: Shiftability values as functions of number of basis functions: (a) $\tilde{r}_{\mathscr{H}_n,1}(\phi)$; (b) $\tilde{r}_{\mathscr{H}_n,\infty}(\phi)$.

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