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2D Wavelet Transforms With A Spatially Adaptive 2D Low Pass Filter

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ABSTRACT

Wavelet transforms that have an adaptive low pass filter are useful in applications where it requires the signal singularities, sharp transitions and image edges to be left intact in low pass subbands. In such applications it is vital to have low pass subbands that are not affected with smoothing artifacts associated with uniform low pass filtering. Previously, we presented a framework for designing 1D wavelets that have a spatially adaptive low pass filter using the prediction first lifting scheme, in which the adaptivity decisions are computed using wavelet (high pass) coefficients and no bookkeeping is required for the perfect reconstruction. In this paper, we extend the 1D scheme to design 2D wavelets that have a spatially adaptive low pass filter. We use 2D polyphase matrix of the corresponding 2D separable transform and compute the 2D lifting factorization steps. This scheme leads to non separable 2D adaptivity decisions, which is the preferred case for images, as opposed to the separable 2D realization using 1D transforms.

1. INTRODUCTION

In recent years, discrete wavelet transforms have become the preferred choice of signal transform in image coding and processing due to efficient decorrelation, energy compaction and the framework for multiresolution analysis associated with the wavelet transforms.

Linear wavelet transforms using filter banks with fixed filter coefficients usually result in uniform smoothing in low pass subbands. This leads to smoothing near edges, sharp transitions and other singularities in signals. Such smoothing artifacts are unacceptable for applications such as transform domain image analysis, transform domain feature extraction and resolution scalable image coding, where the resolution scalability is achieved by retaining only the low pass subband. In such applications it is vital to have low pass subbands that are not affected with smoothing artifacts associated with uniform low pass filtering.

In a previous publication [1] we presented the mathematical framework for 1D wavelet transforms that have an *adaptive* low pass filter that takes into account the underlying signal content using the prediction first lifting scheme [2, 3]. The adaptivity decisions are computed using the wavelet coefficients (high pass subbands) and no bookkeeping is required for the perfect reconstruction.

Lifting has formed a framework for designing adaptive and non linear wavelet transforms [4, 5, 6] The lifting scheme consists of 3 main steps: Split, Prediction and Update. In [4], an adaptive prediction step and in [5, 7], an adaptive update step, leading to adaptive smoothing, have been used in an update first lifting framework, which is different from the classical lifting framework, where the update step follows the prediction step (prediction first lifting). For biorthogonal wavelets, only the prediction first lifting scheme results in a synthesis filter bank wavelet which is longer than the corresponding analysis filter bank wavelet, e.g., 5/3 and 9/7 wavelets. Thus we designed the wavelet transforms that have an adaptive low pass filter in a prediction-first scheme [1], so that the spatially adaptive forms of the traditional wavelet transforms can be easily achieved without affecting the high pass subband coefficients .

In this paper we extend this scheme to design 2D wavelets that have an adaptive low pass filter. Since the 2D polyphase decompositions are used in this case, the 2D adaptivity decisions are made non-separable as opposed to the separable 2D realizations, that use 1D transforms in horizontal and vertical directions. Since the resulting 1D transforms in [1] are non linear, it is not appropriate to consider separable realizations to achieve 2D decompositions. Thus a 2D non separable realization, as proposed in this paper is vital for 2D signal applications. The rest of the paper is organized as follows: In section 2 the 1D scheme is summarized. The design of the 2D scheme is shown in section 3. Examples using the 2D 5/3 wavelet are shown in section 4 followed by the conclusions in section 5.

2. THE 1D SCHEME: REMINDER

In this section we summarize our 1D transform presented in [1]. Let one dimensional input signal x^0 is split into two polyphase components x and y, i.e., $(x, y) = S(x^0)$, where S is an invertible mapping. The channels x, y pass through a two-stage lifting system (P+U lifting) comprising a fixed prediction step using N neighbouring ele-

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ments from x:

$$y'_i = y_i - \sum_{j=1-\lceil N/2\rceil}^{\lfloor N/2\rfloor} p_j x_{i+j} \quad (1)$$

with the prediction weights p_j, where $\sum_j p_j = 1$ and an adaptive update step using $\tilde{N}$ neighbouring elements from y' in the form:

$$x'_i = x_i + \gamma_{d_i} w_i, \quad (2)$$

where $\gamma_d \in [0, \frac{1}{2}]$ and

$$w_i = \sum_{k=-\lfloor \tilde{N}/2\rfloor}^{\lceil \tilde{N}/2\rceil - 1} u_k y'_{i+k} \quad (3)$$

with the update weights u_k, where $\sum_k u_k = 1$. By expanding (3), we obtain:

$$\begin{aligned} w_i &= -\sum_{k=-\lfloor \tilde{N}/2\rfloor}^{\lceil \tilde{N}/2\rceil - 1} \sum_{j=1-\lceil N/2\rceil}^{\lfloor N/2\rfloor} (u_k p_j) x_{i+k+j} \\ &\quad + \sum_{k=-\lfloor \tilde{N}/2\rfloor}^{\lceil \tilde{N}/2\rceil - 1} (u_k) y_{i+k} \quad (4) \end{aligned}$$

The constraints $\sum_j p_j = 1$ and $\sum_k u_k = 1$ yield $\sum_k \sum_j u_k p_j = 1$. Therefore, the value x_i can be subtracted from each term in the right hand side of (4) without any effect to w_i. Hereof, the same summation limits as above are in use unless stated otherwise.

$$\begin{aligned} w_i &= -\sum_k \sum_j (u_k p_j)(x_{i+k+j} - x_i) + \sum_k (u_k)(y_{i+k} - x_i) \\ &= -\sum_k \sum_j (u_k p_j) v_{(x_{i+k+j})} + \sum_k (u_k) v_{(y_{i+k})} \quad (5) \end{aligned}$$

where v is the gradient computed with respected to the element to be updated, x_i. We define the gradient v_{z_l} of an element r_l, where r represents either x or y, with respect to the element x_i as $v_{z_l} = x_i - r_l$.

The weighting parameters $(u_k p_j)$ and (u_k) in (4) correspond to $2(h_{-m}, \ldots, h_{-1}, h_1, \ldots, h_m)$, i.e, twice the corresponding low pass filter coefficients excluding the h_0 coefficient. Thus the quantity w_i represents a weighted mean gradient at element x_i with the low pass filter coefficients excluding h_0 as the weights. Choosing $\gamma_d = \frac{1}{2}$ in (2) yields the linear low pass filter corresponding to original lifting factorization. The value of γ_d can be varied from 0 to $\frac{1}{2}$ in order to vary the strength of the update step, i.e, the low pass filter. For example, a gradient dependent update scheme can be obtained as follows:

$$\begin{aligned} x'_i &= x_i + \gamma_{d_i} w_i \\ d_i &= |w_i| > T \text{ with } \gamma_{0_i} = \frac{1}{2}, \;\; \gamma_{1_i} = 0 \quad (6) \end{aligned}$$

where T is an user defined threshold and γ_0 and γ_1 correspond to the update and no update cases, respectively.

Since the decision d_i at the position i is based only on the weighted gradient mean w_i at i, which is computed using only y' values (refer to (3), which are a priori available to both analysis and synthesis transforms, information regarding adaptivity decision needs not be sent to the synthesis transform. Thus this spatially adaptive low pass filter can be realized with no overheads in bookkeeping.

3. WAVELETS WITH A SPATIALLY ADAPTIVE 2D LOW PASS FILTER

The 2D wavelet transform for 2D data is usually achieved by performing the 1D transform on each dimension as separable processes. We consider the 1D transform matrix F and the 2D signal X of $M \times N$ size to demonstrate the 2D transform operation in order to formulate the framework for the 2D wavelets with an adaptive 2D low pass filter. Assuming the prediction first (P+U) lifting scheme used in the 1D transform the 1D transform matrix F takes the form $\begin{pmatrix} 1-\frac{1}{2}P(z)U(z) & \frac{1}{2}U(z) \\ -P(z) & 1 \end{pmatrix}$ where $P(z)$ and $U(z)$ are the z transform of prediction and update filters. Let F_1 and F_2 be the operation of the transform F in dimensions 1 and 2, respectively. Then the 2D separable decomposition is as follows:

$$\begin{aligned} X'' &= (F_2(X')^t)^t \\ &= (F_2(F_1 X)^t)^t \\ &= F_1 X F_2^t \quad (7) \end{aligned}$$

where ^{t} denotes the transpose of a matrix. By defining the two basis vectors, corresponding to low pass and high pass filters in the 1D transform, in F_1 as F_{10} and F_{11} and similarly the two basis vectors in F_2 as F_{20} and F_{21}, the 2D transform relationship (7) can be rewritten as follows:

$$\begin{bmatrix} X''_a \\ X''_b \\ X''_c \\ X''_d \end{bmatrix} = \begin{bmatrix} \text{vect}(F_{10} \otimes F_{20}^t) \\ \text{vect}(F_{11} \otimes F_{20}^t) \\ \text{vect}(F_{10} \otimes F_{21}^t) \\ \text{vect}(F_{11} \otimes F_{21}^t) \end{bmatrix} \begin{bmatrix} X_a \\ X_b \\ X_c \\ X_d \end{bmatrix} \quad (8)$$

where $\text{vect}\left(\begin{bmatrix} p & r \\ q & s \end{bmatrix}\right) = \begin{bmatrix} p & q & r & s \end{bmatrix}$; the symbol $\otimes$ denotes the outer product;

$$\begin{aligned} F_{10}^t &= \begin{bmatrix} 1-\frac{1}{2}P(z_1)U(z_1) & \frac{1}{2}U(z_1) \end{bmatrix} \quad (9) \\ F_{11}^t &= \begin{bmatrix} -P(z_1) & 1 \end{bmatrix} \quad (10) \\ F_{20}^t &= \begin{bmatrix} 1-\frac{1}{2}P(z_2)U(z_2) & \frac{1}{2}U(z_2) \end{bmatrix} \quad (11) \\ F_{21}^t &= \begin{bmatrix} -P(z_2) & 1 \end{bmatrix} \quad (12) \end{aligned}$$

and $X(z_1, z_2) = X_a(z_1^2, z_2^2) + z_2^{-1} X_b(z_1^2, z_2^2) + z_1^{-1} X_c(z_1^2, z_2^2) + z_1^{-1} z_2^{-1} X_d(z_1^2, z_2^2)$
where the polyphase components: X_a, X_b, X_c and X_d are obtained as follows:

$$\begin{aligned} X_a(z_1, z_2) &= \sum_i \sum_j X(2i, 2j) z_1^{-i} z_2^{-j}, \quad (13) \\ X_b(z_1, z_2) &= \sum_i \sum_j X(2i+1, 2j) z_1^{-i} z_2^{-j}, \quad (14) \\ X_c(z_1, z_2) &= \sum_i \sum_j X(2i, 2j+1) z_1^{-i} z_2^{-j}, \quad (15) \\ X_d(z_1, z_2) &= \sum_i \sum_j X(2i+1, 2j+1) z_1^{-i} z_2^{-j} \quad (16) \end{aligned}$$

where z_1 and z_2 correspond to the dimensions 1 and 2 , respectively. The limits for indices i and j are from 0 to $M/2-1$ and $N/2-1$, respectively. Finally we can summarize the 2D separable wavelet transform as follows:

$$\begin{bmatrix} X''_a & X''_b & X''_c & X''_d \end{bmatrix}^t = F2 \begin{bmatrix} X_a & X_b & X_c & X_d \end{bmatrix}^t \quad (17)$$

where $F2$ is the polyphase component corresponding to the 2D separable transform. Hereof, we use the terms P_1, P_2, U_1 and U_2 to represent $P(z_1), P(z_2), U(z_1)$ and $U(z_2)$, respectively. Thus we show $F2$ using (8)-(17) as follows:

$$\begin{bmatrix} (1-\frac{1}{2}P_1U_1)(1-\frac{1}{2}P_2U_2) & \frac{1}{2}U_1(1-\frac{1}{2}P_2U_2) & \frac{1}{2}(1-\frac{1}{2}P_1U_1)U_2 & \frac{1}{4}U_1U_2 \\ -P_1(1-\frac{1}{2}P_2U_2) & 1-\frac{1}{2}P_2U_2 & -\frac{1}{2}P_1U_2 & \frac{1}{2}U_2 \\ -(1-\frac{1}{2}P_1U_1)P_2 & -\frac{1}{2}U_1P_2 & 1-\frac{1}{2}P_1U_1 & \frac{1}{2}U_1 \\ P_1P_2 & -P_1 & -P_2 & 1 \end{bmatrix} \tag{18}$$

In the above matrix, the products A_nB_m correspond to $\sum_k \sum_j a_k b_j = 1$ when $n = m$ and $A_n \otimes B_m$ when $n \neq m$, where A and B can be either P or U.

3.1. 2D Lifting

Now consider the 2D signal X and its 2D polyphase components: X_a, X_b, X_c and X_d and the polyphase matrix $F2$ of the 2D wavelet transform. We factorize the matrix $F2$ into 4 elementary matrices leading to 4 lifting steps: three 2D prediction steps and a 2D update step. Three prediction steps are as follows:

$$\begin{aligned} X''_d &= X_d \quad -(P_2X_c + P_1X_b - P_1P_2X_a) && (19) \\ X''_c &= X_c \quad -(P_2X_a - \frac{1}{2}U_1X''_d) && (20) \\ X''_b &= X_b \quad -(P_1X_a - \frac{1}{2}U_2X''_d) && (21) \end{aligned}$$

Finally the X_a component is updated using the prediction residuals: X''_b, X''_c and X''_d as follows:

$$X''_a = X_a \quad +\frac{1}{2}\underbrace{(U_1X''_b + U_2X''_c - \frac{1}{2}U_1U_2X''_d)}_{w2} \tag{22}$$

3.2. 2D Adaptive Update Step

In the above 2D lifting factorization, we achieve the prediction steps and the update step analogous to the 1D case. We denote the coefficients of the filters: U_1, U_2, P_1, P_2 , U_1U_2 and P_1P_2 as $u_1(k)$, $u_2(k)$, $p_1(j)$, $p_2(j)$, $(u_1u_2)(l)$ and $(p_1p_2)(l)$, respectively, where $j, k, l \in \mathcal{Z}^2$.

Now we can rewrite the update step in (22) for the update of the point at position $i \in \mathcal{Z}^2$:

$$\begin{aligned} x''_a(i) \;=\; x_a(i) + \frac{1}{2}\Big(&\sum\nolimits_k u_1(k)x''_b(k+i) + \\ &\sum\nolimits_k u_2(k)x''_c(k+i) - \frac{1}{2}\sum\nolimits_l u_1u_2(l)x''_d(l+i)\Big) \end{aligned} \tag{23}$$

Following the above, we can expand the quantity $w2$ in (22) as follows:

$$\begin{aligned} w2(i) \;=\; &\frac{1}{4}\sum\nolimits_{k1}(p_1u_1p_2u_2)(k1)x_a(i+k1) + \\ &-\frac{1}{2}\sum\nolimits_{k2}(p_1u_1)(k2)x_a(i+k2) \\ &+\frac{1}{2}\sum\nolimits_{k3}(p_2u_2)(k3)x_a(i+k3) \\ &+\frac{1}{2}\sum_{k4}u_1(k4)x_b(i+k4) - \frac{1}{4}\sum_{k5}(u_1p_2u_2)(k5)x_b(i+k5) \\ &+\frac{1}{2}\sum_{k6}u_2(k6)x_c(i+k6) - \frac{1}{4}\sum_{k7}(p_1u_1u_2)(k7)x_c(i+k7) \\ &+\frac{1}{4}\sum\nolimits_{k8}(u_1u_2)(k8)x_d(i+k8) \end{aligned} \tag{24}$$

where $k1, \ldots, k8 \in \mathcal{Z}^2$ are the indices for the eight resulting sub filters which form the 2D low pass filter.

The filters $P_1 = P_2 = P$, where P is the 1D prediction filter and $U_1 = U_2 = U$, where U is the 1D update filter. From the 1D lifting constraints, we have $\sum_j p_1(j) = \sum_j p_2(j) = 1$ and $\sum_k u_1(k) = \sum_k u_2(k) = 1$. It can be shown that these constraints yield: $\sum p_1u_1p_2u_2 = \sum p_1u_1 = \sum p_2u_2 = \sum u_1p_2u_2 = \sum p_1u_1u_2 = \sum u_1u_2 = 1$ for the 2D sub filters in (24). Therefore, all the filter coefficients in (24) add up to zero.

Thus we can subtract $x_a(i)$ from each element in the filter template in (24) and compute the gradient value at each element with respect to $x_a(i)$. Then as in the 1D case, the quantity $w2(i)$ represents a 2D weighted mean gradient with the weights corresponding to the resulting 2D low pass filter excluding the (0,0) coefficient. Thus we can compute the 2D adaptive update lifting step as follows:

$$\begin{aligned} x''_a(i) &= x_a(i) + \gamma_r(i)w2(i) && (25) \\ w2(i) &= \sum\nolimits_k u_1(k)x''_b(k+i) + \sum\nolimits_k u_2(k)x''_c(k+i) \\ &\quad -\frac{1}{2}\sum\nolimits_l (u_1u_2)(l)x''_d(i+l) && (26) \end{aligned}$$

where $\gamma_r \in [0, \frac{1}{2}]$. Choosing $\gamma_r = \frac{1}{2}$ in (25) yields the linear 2D low pass filter corresponding to original lifting factorization. The value of γ_r can be varied from 0 to $\frac{1}{2}$ in order to vary the strength of the 2D update step, i.e, the 2D low pass filter. For example, a gradient dependent update scheme can be obtained as follows:

$$\begin{aligned} x''_a(i) &= x_a(i) + \gamma_r(i)w2(i) \\ r(i) &= |w2(i)| > T \text{ with } \gamma_0(i) = \frac{1}{2}, \;\; \gamma_1(i) = 0 \end{aligned} \tag{27}$$

where T is an user defined threshold and γ_0 and γ_1 correspond to the update and no update cases, respectively.

4. AN EXAMPLE

We show an example of choosing the 2D low pass filter adaptively, using the 2D realization of the 1D 5/3 biorthogonal wavelet transform, which is known as (2,2) in lifting terminology. The 1D prediction and update filters are $P = U = (\frac{1}{2}, \frac{1}{2})$. The corresponding prediction lifting steps in 2D considering the four 2D polyphase components: a, b, c and d are as follows:

$$d''_{i,j} = d_{i,j} - \frac{1}{2}(b_{i,j} + b_{i+1,j}) - \frac{1}{2}(c_{i,j} + c_{i,j+1}) + \frac{1}{4}(a_{i,j} + a_{i+1,j} + a_{i,j+1} + a_{i-1,j-1}) \quad (28)$$

$$c''_{i,j} = c_{i,j} - \frac{1}{2}(a_{i,j} + a_{i+1,j}) + \frac{1}{4}(d''_{i,j-1} + d''_{i,j}) \quad (29)$$

$$b''_{i,j} = b_{i,j} - \frac{1}{2}(a_{i,j} + a_{i,j+1}) + \frac{1}{4}(d''_{i-1,j} + d''_{i,j}) \quad (30)$$

where i, j denotes the 2D index. Then the 2D weighted gradient mean $w2_{i,j}$ is computed:

$$w2_{i,j} = \frac{1}{2}(c''_{i,j} + c''_{i-1,j}) + \frac{1}{2}(b''_{i,j} + b''_{i,j-1}) + -\frac{1}{4}(d''_{i,j} + d''_{i-1,j} + d''_{i,j-1} + d''_{i-1,j-1}) \quad (31)$$

and the binary decision update lifting step is as follows:

$$\begin{aligned} a''_{i,j} &= a_{i,j} + \gamma_{r_{i,j}} w2_{i,j} \\ r_{i,j} &= |w2_{i,j}| > T \text{ with } \gamma_0 = \frac{1}{2},\ \gamma_1 = 0 \end{aligned} \quad (32)$$

The inverse transform is obtained by operating the above lifting steps in the reverse order, i.e., (32)-(28), and using opposite signs. Similar factorization can be performed using other biorthogonal wavelets in order to achieve the 2D wavelets yielding spatially adaptive 2D low pass filters.

We show the decision maps for the 2D 5/3 spatially adaptive low pass filter in Fig. 1, where the spatial points correspond to the decision $r_{i,j} = 1$, i.e., the points correspond to the edges are shown in black. With this proposed transform, a low pass filter with a low update strength leading to higher low pass bandwidth is used centered on such points. In Fig. 2, we show the low pass subbands after one decompoistion level for 5/3 wavelet and for the corresponding transform yielding the spatially adaptive 2D low pass filter. It is evident from Fig. 2 that the proposed transform preserves edges in the low pass subband.

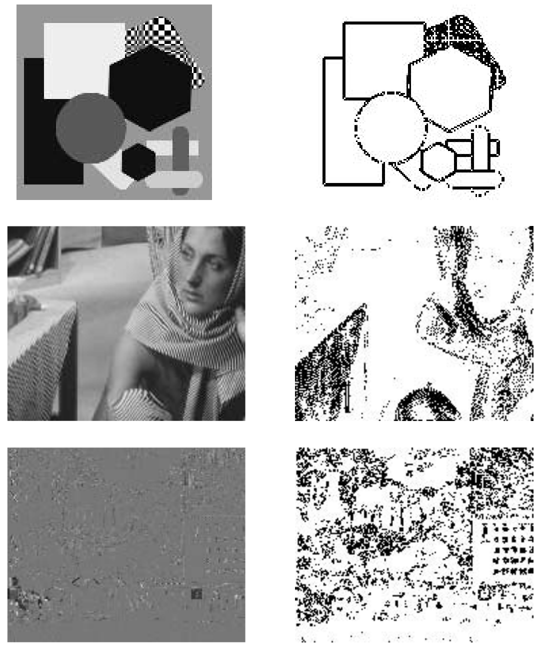

Fig. 1. Decision maps. Column 1: The polyphase a of test images: a synthetic image, a natural image and a residual frame from video coding; Column 2: decision maps($r_{i,j} = 1$ in black and $r_{i,j} = 0$ in white) for scale 1.

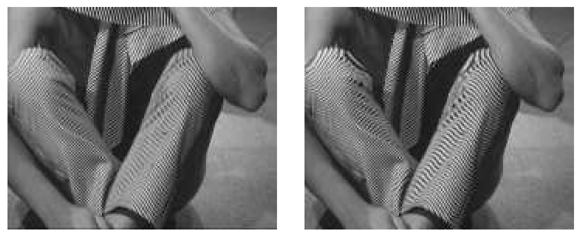

Fig. 2. Half resolution images: using the linear 2D 5/3 wavelet *(left)* and using the 2D 5/3 wavelet yielding a spatially adaptive 2D low pass filter *(right)*.

5. CONCLUSIONS

In this paper, a framework for 2D wavelet decompositions that have an adaptive 2D low pass filter was developed using lifting. The adaptivity criterion, based on a 2D weighted mean gradient computed using the wavelet coefficients in high pass sub bands in the same scale, can be recovered in the synthesis without any bookkeeping. This scheme is useful in image/video coding with resolution scalability functionality, image sub sampling and transform domain image analysis and feature extraction.

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