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A New, Fast and Low-Cost FFT Estimation Scheme of Signals Using 1-Bit Non-Subtractive Dithered Quantization

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ABSTRACT

This paper studies the use of coarse quantization schemes, and the exploitation of their associated practical advantages, in the computation of one of the cornerstone algorithms in digital signal processing (DSP), i.e., the FFT algorithm. Such advantages include low structural complexity, low implementational cost and high computational speed. However, as the quantization resolution gets smaller, the irreversible loss in computational accuracy becomes more prohibitive, thus precluding the use of these attractive practical advantages. We propose here a new theory which will allow the use of any coarse quantization scheme, including the crudest possible (i.e. 1-bit) while, in practice, incurring only a negligibly small loss in accuracy. The proposed theory hinges on the use of dithered quantization scheme. The 1-bit version of this theory has been successfully tested on simulated and real signals, including music and speech signals, as evidenced by our simulation results. Finally, these results provide a strong encouragement to extend this theory to noisy signals as well as to numerous other important transforms.

1. INTRODUCTION

Fourier transformation is undoubtedly one of the corner stones of digital signal processing (DSP). The breadth of its applications is such that it transcends the confines of DSP, reaching out to several scientific and engineering fields as varied as numerical analysis, image processing, filtering, communications, radar, sonar and seismic signal processing. The discovery of the now-famous radix-2 fast Fourier transform (FFT) algorithm [1] has transformed the discrete Fourier transform (DFT) from being almost a mere "academic" curiosity, with limited applications, to being a powerful computational tool whose applications continue to grow unabatedly. The original FFT algorithm underwent several structural modifications that were successfully implemented [2]. A fairly recent review of the state-of-the-art of FFT algorithms can be found in [3,4].

The numerous variations of the original radix-2 FFT algorithm were brought about through the dual use of the exploitation of symmetry properties inherent in the FFT algorithm and the principle of "divide and conquer". However, both the original FFT algorithm and all its existing variants rely, in their digital implementation, on input signals that are sufficiently highly quantized (i.e. resolution $\geq$ 8 bits). Although low quantization resolutions entail an irreversible loss of accuracy which becomes more prohibitive as the resolution gets smaller, they nevertheless offer several practically attractive advantages primarily associated with the use of shorter word lengths. Such practical advantages include structurally simple, low-cost and fast FFT processing schemes that can only enhance the already-high speed boasted by existing FFT algorithms. To unlock these potential practical advantages, a way to reconcile two seemingly disparate requirements, namely achieving a high accuracy in processing the FFT of a given signal while at the same time using only a coarsely-quantized version of this signal, has to be found. The main contribution of this paper is therefore to offer a new solution to this important problem based on a new theory of exact-moment recovery (EMR). This theory hinges upon the use of a simple signal-coding scheme based on the non-subtractive dithered quantization (NSDQ) technique [5]. With this new scheme, it is shown that even when the crudest possible (i.e. 1-bit) NSDQ technique is used, the computational accuracy of the FFT is only negligibly impaired. Also, the loss of accuracy remains acceptable even in moderately noisy environments. The EMR theory was also successfully tested in other areas such as higher-order statistics [6] and frequency response estimation [7]. The 1-bit NSDQ-based computational scheme studied in this paper is diagrammatically depicted by Figure 1 and is applied to one channel (input channel) only. As seen from Figure 1, the NSDQ quantizer is basically equivalent to a classical quantizer with a modified (in this case dithered) input. As such, the use of a 1-bit NSDQ quantizer on the input reduces the multiplication of the input by the 2 basis functions involved in the FFT algorithm, to a modified relay -type of operation. Therefore the 1-bit NSDQ-based FFT scheme studied in this paper will henceforth be referred to as the modified relay FFT (or MR-FFT, for short). The proposed one-channel 1-bit NSDQ-based FFT estimation has been successfully tested on both music and speech signals.

2. A KEY NEW THEOREM ON THE NSDQ - BASED SINGLE CHANNEL ESTIMATION OF THE DFT/FFT

This section first reviews some basic definitions used in the EMR theory, including the definition of the NSDQ quantization scheme. It then states a new key Theorem, first in its general form upon which hinges the main contribution of this paper, and then in its frequency-domain first-order form (Theorem 1) which offers directly the desired solution of an accurate 1-bit FFT estimation scheme.

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2.1 Definition Of The NSDQ Quantization Scheme

Given an input x and a (user-defined) dither signal D that is statistically independent of x, then a non-subtractively dithered quantization (NSDQ) of x is equivalent to the classical quantization (Q) of the dithered signal $y = x + D$, i.e.

$$x \rightarrow x_{NSDQ} = NSDQ(x) = Q_a(y) = y_Q \qquad (1)$$

Here, Q_a represents the entire class of uniform classical quantizers parameterized by the step (q) and the shift factor given by $a \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, i.e.:

$$y_Q = \left(a + n + \frac{1}{2}\right) q \qquad if \quad y \in \left[(a+n)q, (a+n+1)q\right] \qquad (2)$$

2.2 Definition Of The p^{th} Order Class Of Linearizing Dither Signals $\boldsymbol{D}_p$

Given an ergodic and stationary dither signal D and its characteristic function $W_D(u)$, then:

$$D \in \mathrm{D_P} \Leftrightarrow W_D^{(r)}\left(\frac{2n\pi}{q}\right) = 0, \forall\, r \in [0, p-1] \; and \;\; n \neq 0 \qquad (3)$$

According to the closure property of $\boldsymbol{D}_p$ [4], we can say that if $D \in \boldsymbol{D}_p$ and for any signal x that is statistically independent of D, then the dithered signal $y = (x + D) \in \boldsymbol{D}_p$.

2.3 A Key First-Order Theorem And Its Application To The Exact Recovery Of The FFT Based On The MR-FFT Estimator

First let us recall from [6] that the NSDQ quantizer is characterized by its p^{th}-order ($p \geq 1$) moment-sense input/output function (MSIOF), $h_p(x)$, given by:

$$h_p(x(n)) = \sum_{k=0}^{p} c_k x^k(n) \quad where$$

$$c_k = \sum_{t=0}^{p-k} \left\{ \frac{p!}{(p-k-t+1)!\; k!\; t!} \left(\frac{q}{2}\right)^{p-k-t} \cdot E\left[R^t\right]\left[p \oplus k \oplus t \oplus 1\right] \right\} \qquad (4)$$

with $\oplus$ *denoting modulo-2 operation.*

In this paper, we only consider the first-order case, when $p = 1$. A higher-order case will be reported later.

2.3.1 *Theorem 1 (First-Order Case):*

Given a 1-D NSDQ quantizer whose MSIOF, input, output and dither signals are given respectively by $h_p(x), x, x_{NSDQ}$, *and* D *where* D *is both zero-mean and statistically independent of* x, *and given that* $X(\omega_i) = DFT[x(n)]$ *and* $X_{NSDQ}(\omega_i) = DFT\left[x_{NSDQ}(n)\right]$ $\forall\, i, n \in [1, N]$, *then:* ***(a)*** $X_{NSDQ_p}(\omega_i)$ *and the first-order frequency-domain polynomial mapping, defined by* $H_1(\omega_i) = DFT\left[h_1(x(n))\right]$, *are moment-sense equivalent and hence* ***(b)*** $X(\omega_i)$ *can be exactly recovered from the expected value of* $X_{NSDQ}(\omega_i)$, *i.e.:*

$$E\left[X_{NSDQ_p}(\omega_i)\right] = E\left[H_1(\omega_i)\right] = X(\omega_i) \qquad (5)$$

Note here that since $h_1(x(n))$ gives an average I/O function of the NSDQ quantizer, its DFT, $H_1(\omega_i)$, gives the equivalent average frequency-domain characterization of the NSDQ quantizer.

2.3.2 *Application of Theorem 1 to the MR-FFT estimation*

Part **(a)** of Theorem 1 can be proved using the EMR theory of [6]. For part **(b),** it can be easily shown that for p=1, (4) reduces to the perfectly linear 1^{st}-order (i.e. average) MSIOF of the NSDQ quantizer, i.e. $h_1(x(n)) = x(n)$. Hence, $H_1(\omega_i) = DFT\left[(x(n)\right]$, which leads directly to the desired equality of (5), i.e. the FFT of the unquantized signal $x(n)$ can be exactly recovered from the FFT of its binary version.

3. SIMULATION AND CONCLUSION

In order to assess the validity of the proposed 1-bit NSDQ-based FFT estimation theory, three different simulations were run for the FFT spectrum evaluation. The first example involves a single sinusoid and is used primarily to demonstrate the estimation accuracy of the proposed NSDQ scheme when compared to the classical undithered quantization. In the second example, the proposed MR-FFT estimator is used to estimate the spectrum of a musical recording. As a final test, the MR-FFT estimator is used to analyze the sound signal of utterance of the word *"Matlab"*. The block diagram of the simulation setup is given in Fig.1. A detailed description of the simulation work is as follows:

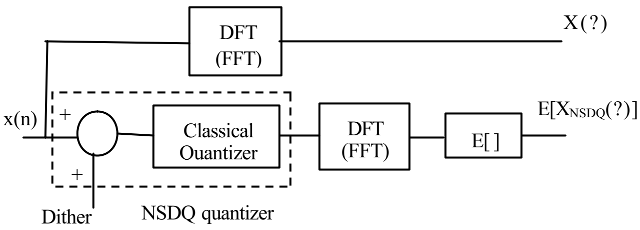

Fig.1: NSDQ quantization-based FFT Estimation (MR-FFT)

A sinusoidal signal of amplitude $A=10$ and frequency $f=500\ Hz$ is used as the input signal $x(n)$. A total of 6400 points are used for the estimation of the FFT magnitude spectrum. Fig. 2 shows the amplitude spectra of the original signal (non-quantized) and the non-dithered 1-bit quantized signal. The two spectra show that **(a)** there is a large difference between the two spectra magnitudes at the test frequency (resulting in an FFT magnitude estimation error of about 35%) and **(b)** there is a noticeable presence of spurious signal peaks, in the non-dithered quantized spectrum, at frequencies other than the test frequency, thus resulting in a well-structured additional error which increases the total error.

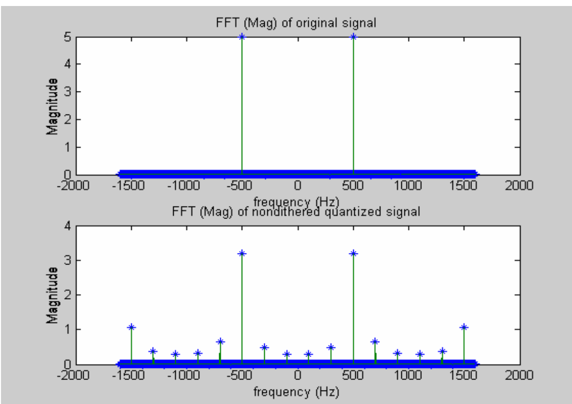

Fig.2: FFT (Mag) spectra of an unquantized and a 1-bit NSDQ-quantized single frequency signal

In the next simulation, a zero-mean random dither signal and the input signal, both of equal peak-to-peak amplitude, are added together and their sum is then 1-bit quantized. Next the FFT magnitude spectrum of this 1-bit quantized sum signal is computed using the proposed scheme. The results, shown in Fig.3, clearly demonstrate that the MR-FFT estimator has not only fully recovered the FFT magnitude spectrum, with an error in the range of 5%, but has also replaced the structured harmonics-related error by a random-like (i.e. unstructured) error pattern. The magnitude of this random error is almost 5% of the peak magnitude spectrum. Increasing the number of samples will lead to a further reduction in this relative error.

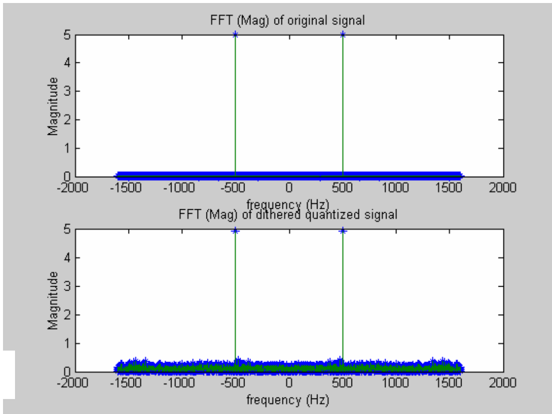

Fig.3: FFT (Mag) spectra of an unquantized and a 1-bit dither-quantized single frequency signal

A clarinet tune is selected as a practical example to test the performance of the MR-FFT estimator on a real signal. The tune was recorded at a quantization resolution of 16 bits per sample. The sampling frequency was 16,000 samples per second and the duration was 1.8 seconds. As the estimation accuracy increases with the total number of samples processed, the limited record of the clarinet tune data was replicated a number of times to increase the total number of samples to be processed. This augmented data set was then used in the proposed 1-bit dithered quantization scheme. The results, shown in Fig. 4, show the excellent recovery of the original magnitude spectrum of the recorded tune by the proposed scheme. The level of the noisy-pattern at the baseline in Fig. 4 can be further attenuated by processing more samples. As a further test of the excellent FFT spectrum recovery capability of the proposed scheme, an approximate value of the peak of this random noise floor was subtracted from the FFT magnitude spectrum of Fig. 4 and the inverse-FFT of the resultant magnitude spectrum was computed. The reconstructed sound was played back and the hearing results were very encouraging and thus very supportive of the proposed scheme.

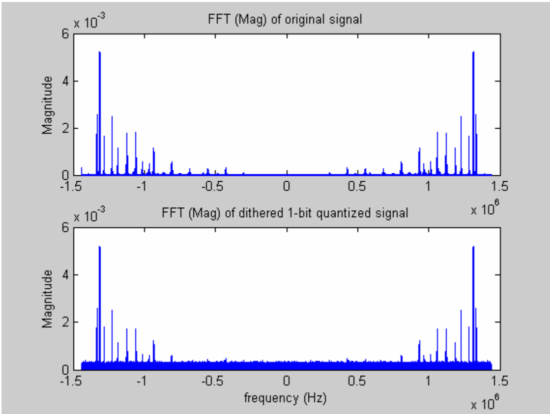

Fig.4: FFT (Mag) spectra of an unquantized and 1-bit Dither-quantized Clarinet tune

In our last example, a sound recording of the utterance ***"Matlab"*** is used as a test signal for the proposed scheme The sound recording was saved at a resolution of 16 bits per sample using a sampling frequency of 8 KHz. Fig. 5 below shows the FFT magnitude spectra of the unquantized and the 1-bit NSDQ-quantized sound signals. Here again, the simulation results of Fig. 5 demonstrate the excellent FFT magnitude spectrum recovery achieved by the proposed scheme. As before, the noise floor in Fig. 5 can be further reduced through the processing of more samples.

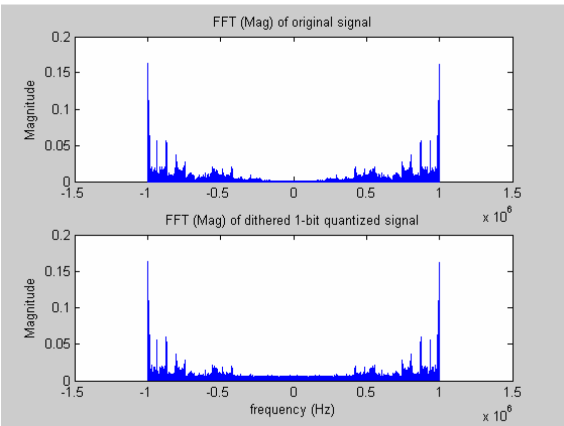

Fig.5: FFT (Mag) spectra of an unquantized and 1-bit Dither-quantized utterance "***Matlab***"

In conclusion, it can be said that the excellent results, obtained here on both simulated signals as well as recordings of real signals, provide ample encouragement to extend the application of this practically-important 1-bit FFT estimation technique, not only to noisy signals but also to the accurate, fast and low-cost estimation of a variety of other transforms.

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4. REFERENCES

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