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AN EASILY IMPLEMENTABLE SAMPLING PROCEDURE FOR CERTAIN FRACTAL AND OTHER NON-BAND LIMITED SIGNALS

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ABSTRACT

The first step in digital signal processing is analog-to-digital conversion. Classically, the analog signal is sampled via the process described by the Whittaker-Kotelnikov-Shannon theorem (WKS), which requires that the original signal is band-limited. Irregular signals, as are fractal ones, typically do not verify such an assumption. The low-pass filtering necessary to apply the WKS theorem implies then a serious loss of information. Although extensions of the WKS theorem to non-band limited signals do exist, their implementation is typically rather difficult. We propose in this paper a sampling procedure for some classes of non-band limited signal which shares the simplicity of the WKS procedure (involving only bandpass analog filtering and regular sampling). *We give a detailed explanation of the implementation of our sampling scheme, and provide some of its theoretical properties. Numerical experiments illustrate the behaviour of the method.*

1. INTRODUCTION AND BACKGROUND

Sampling theory is a rich and important field in signal processing. The basic theorem is the Whittaker-Kotelnikov-Shannon theorem (WKS), which ensures perfect reconstruction of band-limited signals through regular sampling at a sufficiently high frequency. Classical extensions include the case of non-regular sampling, or sampling in more general spaces of functions. See [4, 5, 7] for excellent reviews and extensions.

The case of non-band limited signals has been considered by a number of authors. Dealing with this situation is important since many real-world signals are for instance time-limited or irregular (i.e. not C^∞), and thus do not belong to the set of band-limited signals. Among many approaches, let us mention the use of bases or transforms other than the Fourier one [7], or the theory of shift invariant spaces [1, 8]. Though such approaches provide satisfactory theoretical results, they lack the practical simplicity of the WKS theorem. Indeed, it is not clear how to actually implement such schemes using simple circuits and filters.

Our aim in this work is to provide an easily implementable procedure for the sampling of certain classes of non-band limited signals: Our sampling scheme can be realized using only bandpass analog filtering and regular sampling, as in the case of the WKS theorem for band-limited signals. As in many other approaches, our scheme allows to approximate any signal in a large functional space with control of the reconstruction error.

The paper is organized as follows: In section 2, the signals class and sampling procedure are described, and the main result of reconstruction is given. Section 3 describes the proof of the main ingredient used in the reconstruction theorem. Various results pertaining to the errors entailed by a wrong estimation of the parameters are grouped in section 4. Numerical experiments are displayed in section 5.

2. SAMPLING PROCEDURE AND MAIN RESULTS

2.1. The space of signals

We start by recalling the definition of slowly varying functions [2]:

Definition 1 (Slowly Varying functions). *A function h is said to be slowly varying at $+\infty$ if*

$$\forall\lambda > 0,\ \lim_{x\to+\infty}\frac{h(\lambda x)}{h(x)} = 1$$

A strong result about these functions is [2]:

Proposition 2. *If h is slowly varying at $+\infty$, then for $0 < u < v$,*

$$\lim_{x\to+\infty}\frac{h(\lambda x)}{h(x)} = 1$$

uniformly in $\lambda \in [u; v]$

Definition 3 (Admissible family). *Let Ω be a subset of $\mathbb{R}^n$. The parametric family $F = \{S_\beta\}_{\beta\in\Omega}$ of functions of $L^2(\mathbb{R})$ is said to be admissible if*

- $\forall\beta \in \Omega, \lim_{|f|\to+\infty} \widehat{S}_\beta(f) = 0$
- $\forall\beta \in \Omega, \widehat{S}_\beta \circ \log$ *is slowly varying at $+\infty$.*

- *Given $0 < u < v$ and $z \in \mathbb{C}$, the solution in β of the equation $\widehat{S}_\beta(u) = z\widehat{S}_\beta(v)$ is unique (and can be easily computed).*

For concreteness, it is useful to exhibit examples of admissible families. The two following families are of special interest, both from a theoretical point of view and in applications.

- *Exponential family:* $S_\beta(t) = e^{-\beta|t|}$, with $\Omega = \mathbb{R}_+^*$.
- *1/f family:* $\widehat{S}_\beta(f) = \frac{\chi_{|f|>1}}{|f|^\beta}$ with $\Omega = (1/2, \infty)$.

Note that the theory also holds if β is multivariate, for instance $\beta = (\beta_1, \beta_2)$ with $\widehat{S}_\beta(f) = \chi_{|f|>1} e^{-\beta_1 |f|^{\beta_2}}$ and $\beta_1 > 0$, $0 < \beta_2 < 1$. Given an admissible family of functions, we can now construct our space of signals.

Definition 4 (Space of reconstructible signals). *Let F be an admissible family, f_c be a positive real and $PW(f_c)$ denote the set of functions of $L^2(\mathbb{R})$ whose Fourier transform is compactly supported in $[-f_c; f_c]$. Define the functional space:*

$$\mathcal{E}(F, T, f_c) = (PW(f_c) \bigoplus \{\sum_{k\in\mathbb{Z}} c_k S_\beta \left(\frac{t-kT}{T}\right)$$

$$\textit{such that } \{c_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) \textit{ and } \beta \in \Omega\}$$

Interpreting this definition in the Fourier domain, we see that we consider the space of functions X in $L^2(\mathbb{R})$ such that if $|f| > f_c$, $\widehat{X}(f) = \widehat{g}(Tf)\widehat{S}_\beta(Tf)$ for some $\beta \in \Omega$, where $\widehat{g}(f) = \sum_k c_k e^{-2i\pi kf}$ is a periodic function.
In our approach, f_c and T are supposed to be known. This is not a practical restriction since in the bandlimited sampling framework, the Nyquist frequency is assumed to be known as well. In contrast, β is not assumed to be known. This is crucial, since β will often be the important parameter that characterizes $\mathcal{E}(F, T, f_c)$. Let us consider for instance the case of the 1/f family. The associated space $\mathcal{E}(F, T, f_c)$ is composed of "fractal" signals, in the sense that their Fourier spectrum has an envelop decaying as $1/f^\beta$, with the parameter β describing the fractal properties of signals in this class. For our approach to be useful in the frame of fractal signal processing, it is important to cope with the fact that β is in general unknown *a priori*. Our method allows to estimate β in the course of the sampling procedure.
Since we can always process low frequencies through the WKS theorem, we shall assume in the following that we have removed the low frequencies. More precisely, we can assume that $\widehat{S}_\beta(f) = 0$ if $|f| \leq f_c$ and focus on the following space:

$$\left\{\sum_{k\in\mathbb{Z}} c_k S_\beta \left(\frac{t-kT}{T}\right) : \{c_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}) \text{ and } \beta \in \Omega\right\}$$

To be able to recover a signal $X(t) = \sum_k c_k S_\beta \left(\frac{t-kT}{T}\right)$, we shall compute $\{c_k\}_{k\in\mathbb{Z}}$ and β from the analog signal X. If β is known, the natural way to compute c_k with infinite precision is to take the inner product of of X with the dual basis $\{\widetilde{S}\left(\frac{t-kT}{T}\right)\}$ associated to S_β. This comes down to filtering $X(t)$ with the adapted ideal filter $\widetilde{S}_\beta(-t)$ and then to sampling it at pace T. This is difficult in practice, since one has to implement a *continuous* filtering with some filter that may be very complex. Moreover, in our more general problem, β *is unknown*.

2.2. Sampling procedure

In the following, we assume that $T > 0$ is known. We implement the procedure described on figure 1 to compute a discrete family $\{I_k(a)\}$.

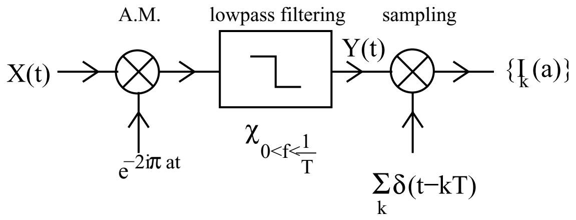

Fig. 1. Sampling procedure. Note the first two cells (amplitude modulation+lowpass filtering) can be implemented as a band-pass filtering

The steps involved may be described as follows:

- We first perform an amplitude modulation (A.M.) on the analog signal $X(t)$ by multiplying $X(t)$ by $e^{-2i\pi at}$ for some $a > 0$ large enough (we will give a precise meaning to this later on).
- Then we low-pass filter the resulting signal, keeping only frequencies between 0 and $\frac{1}{T}$. Note that these first two steps can be performed by bandpass filtering with band $a < f < a + \frac{1}{T}$.
- The obtained (analog) signal $Y(t)$ is finally sampled at constant pace T

Before going further, let's notice that this procedure is easily implementable since it only involves simple signal processing operators that can be realized with usual circuits.
The main idea that allows to estimate the parameters characterizing the signal is that, for $a > 0$ large enough, $I_k(a) \simeq \widehat{S}(Ta)e^{-2i\pi kaT} c_k$ (this fact shall be proved later). This means that the family $\{e^{2i\pi kaT} I_k(a)\}$ is proportional to the family $\{c_k\}$, and that the factor of proportionality contains information on β.

To compute β, we need an additional coefficient, for instance $I_k(a+u)$ (where $u > 0$) and we solve the equation

$$\widehat{S}(Ta) = \widehat{S}(Ta+Tu)\frac{I_k(a+u)}{I_k(a)}e^{-2i\pi uT}$$

Remark 5. *In most cases of interest, $\widehat{S}_\beta$ is a positive real function as it is supposed to model some frequency envelop. As a consequence, one can get rid of the exponential factor in the above equation by taking the moduli of each side:*

$$\widehat{S}(Ta) = \widehat{S}(Ta+Tu)\left|\frac{I_k(a+u)}{I_k(a)}\right|$$

Remark 6. *Increased robustness is obtained by estimating $I_k(a)$ for several values of a and using a regression.*

Since all the information characterizing the signal $X(t)$ lies in $\{I_k(a)\}_{k\in\mathbb{Z}} \cup \{\beta\}$, one may finally reconstruct X from these coefficients using the formula:

$$X(t) = \sum_{k\in\mathbb{Z}} c_k S_\beta\left(\frac{t-kT}{T}\right)$$

3. MAIN RESULT

In the following, $\widehat{g}(f) = \sum_k c_k e^{-2i\pi kf}$ belongs to $L^2([0;1])$ and we note

$$\|\widehat{g}\|_2 = \left(\int_0^1 |\widehat{g}(f)|^2 df\right)^{\frac{1}{2}}$$

Interpreting figure 1 leads to

$$I_k(a) = T\int_0^{\frac{1}{T}} \widehat{X}(f+a)e^{2i\pi kfT} df$$

We have claimed that for $a > 0$ large enough, $I_k(a) \simeq \widehat{S}(Ta)e^{-2i\pi kaT}c_k$. More precisely (see [6] for a proof),

Proposition 7. *If $\{S_\beta\}_{\beta\in\Omega}$ is an admissible family of functions, then*

$$\left|I_k(a) - \widehat{S}_\beta(Ta)e^{-2i\pi kaT}c_k\right| \leq \|\widehat{g}\|_2\varepsilon(a)\widehat{S}_\beta(Ta)$$

where $\varepsilon(a)$ tends to zero when a tends to $+\infty$.

4. ROBUSTNESS

We now study the robustness of our sampling method to errors committed on the different parameters of the problem, i.e. T, β, and $I_k(a)$. For this to be done, we need to assume some regularity on $\widehat{g}$ and $\widehat{S}_\beta$, and we will make the following assumptions:

$$\widehat{g}' \in L^q([0,1]) \text{ for some } q \geq 1 \tag{1}$$

$$\frac{\partial \widehat{S}_\beta}{\partial \beta} \in L^\infty(\mathbb{R}) \tag{2}$$

4.1. Robustness with respect to an error on β

In this section we suppose that an error is made on the estimation of β and we want to obtain an estimate on the error of reconstruction of X. We thus suppose that the true value of β is β_0, but we reconstruct X as

$$\widetilde{X}(t) = \sum_{k\in\mathbb{Z}} c_k S_{\beta_1}\left(\frac{t-kT}{T}\right)$$

where $\beta_1 = \beta_0 + \delta\beta$.
A concept necessary to our study is that of a Riesz basis:

Definition 8 (Riesz basis [3]). *A family of functions $\{\phi(t-k)\}_{k\in\mathbb{Z}}$ is said to be a (shift-invariant) Riesz basis if and only if there exist $B > A > 0$ such that for all sequence $\{c_k\}_{k\in\mathbb{Z}} \in l^2$,*

$$A\sum_{k\in\mathbb{Z}}|c_k|^2 \leq \left\|\sum_{k\in\mathbb{Z}} c_k\phi(t-k)\right\|_2^2 \leq B\sum_{k\in\mathbb{Z}}|c_k|^2$$

Shift-invariant Riesz bases are characterized as follows [3]:

Proposition 9. *$\{\phi(t-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis if and only if there exist $B > A > 0$ such that*

$$A \leq \sum_{k\in\mathbb{Z}}|\widehat{\phi}(f-k)|^2 \leq B \text{ a.e.} f$$

One may then prove the following result [6]:

Proposition 10. *If $\left\{\frac{\partial S_\beta}{\partial\beta}(t-k)\right\}$ is a Riesz basis, and if for almost all f, $\frac{\partial\widehat{S}_\beta}{\partial\beta}(f)$ is a monotonic function of β, then there exists $B > 0$ such that*

$$\|\widetilde{X} - X\|_2 \leq B\|\widehat{g}\|_2\delta\beta$$

It is easily checked that the exponential and the $1/f$ families verify the assumptions of proposition (10).

4.2. Robustness with respect to an error on $\{c_k\}_{k\in\mathbb{Z}}$

If one assumes that an error is committed on the estimation of the coefficients $\{c_k\}$, then the reconstruction will be imperfect for two reasons: first we reconstruct using coefficients $\{\widetilde{c}_k\}$ different from the true coefficients $\{c_k\}$, and second the estimation of β relies on the estimation of these coefficients, and therefore will not be accurate.
The second source of error has been studied in the previous section. We focus here on the consequence of an error made on the coefficients. Let $\widetilde{c}_k$ denote the coefficients of the reconstructed signal $\widetilde{X}$.

Proposition 11. *If $\{S_\beta(t-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis then there exists $B > 0$, independent of $\{c_k\}$ and of $\{\widetilde{c}_k\}$ such that*

$$\|\widetilde{X} - X\|_2 \leq B\sum_{k\in\mathbb{Z}}|\widetilde{c}_k - c_k|^2$$

4.3. Robustness with respect to an error on T

From the very beginning, we have supposed that the value of T was known. We investigate here the situation where the knowledge of T is subject to some error δT. Thus we set $T' = T + \delta T$ and we assume that the procedure described in section 2 is applied with T' instead of T. The next proposition shows that the family $\{e^{2i\pi kTa} I_k(a)\}$ so obtained remains proportional to the family $\{c_k\}$ up to a first order approximation.

Proposition 12. *If the algorithm is applied with $T' = T + \delta T$ instead of T, then*

$$\left| I_k(a) - \widehat{S}_\beta(Ta) e^{-2i\pi kaT} c_k \right| \leq \left| \widehat{S}_\beta(Ta) \right| \left(\varepsilon(a) + \frac{|\delta T|}{T'} \right)$$

where $\varepsilon(a)$ tends to zero as a tends to $+\infty$.

Thus the error is uniformly bounded (in k) by a term that depends both on the value of a and that of the relative error on T. The proof of this proposition may be found in [6].

Remark 13. *The factor of proportionality $\widehat{S}_\beta(Ta)e^{-2i\pi kTa}$ is unknown since it depends on T. This is a problem for the computation of β. However, in the case of interest $\widehat{S}_\beta(f) = \frac{1}{|f|^\beta}$ for $|f| > f_c$, then $\log(|I_k(a)|) = -\beta \log(|a|) - \beta \log(T) - \log(|c_k|) = -\beta \log(|a|) + \gamma$ where γ does not depend on a. Thus, it is clear that by computing $I_k(a_i)$ for different values of $a = a_1, ..., a_n$, one is able to estimate β. Likewise, one can improve our knowledge of T by looking at the phase of $I_k(a)$ for several values of $a = a_1, ..., a_n$: $\arg(I_k(a)) = 2\pi k a_i T \mod 2\pi$ and performing a regression.*

5. NUMERICAL EXPERIMENTS

We present numerical experiments using the family $S(t) = e^{-\beta|t|}$, with $\beta = 5$. The case where $\widehat{S}_\beta(f) = \frac{1}{|f|^\beta}$ leads to similar results. The numerical experiment consists in applying the procedure described in figure 1. The results shown on figure (2) are thus exactly the ones that would be obtained in real situations using only simple filters. As one can see, the reconstructions are indistinguishable from the original.

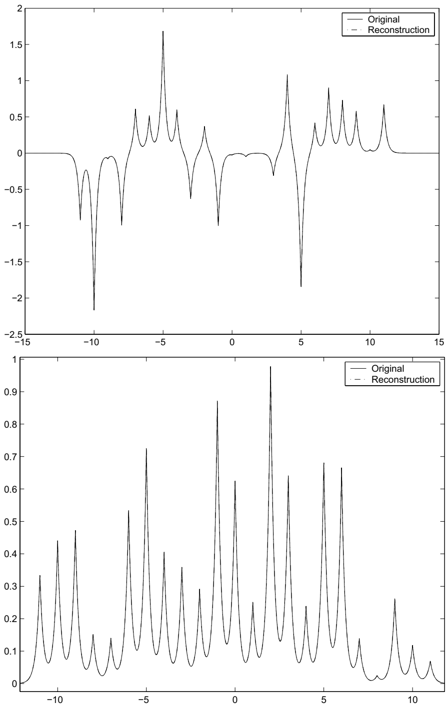

Fig. 2. Two examples of original and reconstructed signals with $\beta = 5$. The c_k are chosen in a random way. The estimated β are respectively 4.98 (top example) and 4.97 (bottom example).

6. REFERENCES

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