ARE BLIND EQUALIZERS ASSUMING WHITE DATA APPLICABLE TO BLOCK ENCODED SEQUENCES?

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ABSTRACT

Several blind channel equalization/identification algorithms [8, 6, 7] assume the transmitted information sequence to be white. In real communication systems, redundancy is added to the source sequence in order to be able to detect and correct symbol errors in the receiver. It is not obvious how channel encoding will affect the assumption of whiteness. We analyze the autocorrelation sequence of linear block codes. The codes are presented in terms of a Markov model, for which the autocorrelation can be analytically obtained. The encoded data is used in the prediction-based blind equalizer of [8] and the performance is empirically compared to the case of unencoded data.

1. INTRODUCTION

A blind channel equalization/identification algorithm attempts to mitigate the effects of the Inter-Symbol Interference (ISI) introduced by the effective communication channel. Because no known training signal is available, some statistical criteria need to be used. For a number of closely related blind equalizers [8, 6, 7], the transmitted sequence $x(n)$ is assumed white, i.e. the autocorrelation function $R_x(j) = \mathrm{E}[x(n)x^*(n-j)] = \delta(j)$. This is a reasonable assumption for efficiently compressed data. However, in a channel encoder, the data sequence is augmented with parity bits for the purpose of error detection and correction. In this paper, we study the effect of the channel encoding to the autocorrelation function. We demonstrate that block encoded sequences are white for most practical codes. We will assume the source vector to be a Bernoulli sequence. However, the analysis can be performed for any source that can be modeled as a Markov chain [1].

In linear block encoding, a block $\mathbf{a}_M$ of M L-ary symbols from the source sequence $a(n)$ is represented by a length-N L-ary code word $\mathbf{c}_i, i = 0, \ldots, K-1$, where the number of all code words $K = L^M$. The code words are then used to construct the eventually transmitted sequence $x(n)$. The L-ary alphabet can be mapped to k bits, if $L = 2^k$, resulting in a binary code. A block of a consistent code consists of two parts, the M information symbols unchanged appended with a parity check pattern that is computed from the M symbols. The code word can be generated by a linear shift register, as in Fig. 1, where $M = 4$ and $N = 7$ [9]. After the information symbols have been fed through as they are, the switch is turned to the parity generating part and the parity symbols are then concatenated to the information bits. In this paper, the bits of the code word are mapped to the transmitted sequence $x(n)$ as $0 \rightarrow -1$ and $1 \rightarrow +1$.

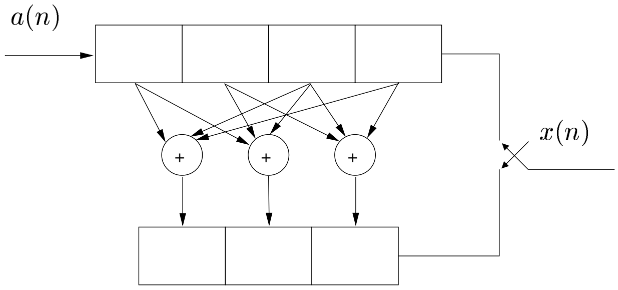

Figure 1: A shift register to produce a linear block code [9].

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2. AUTOCORRELATION FUNCTION OF A CHANNEL ENCODED SEQUENCE

In order to analyze the autocorrelation function of the encoded sequence, we model it as a Markov chain. Such modeling has been considered in [2, 4, 1]. We consider a *Mealy machine*, so that the output codeword is a function of the current state only. When the Markov model is in state $s_i, i = 0, \ldots, K-1$ at time instant n, the corresponding output code word of length N is $\mathbf{c}_i = [c_i(0), \ldots, c_i(N-1)]$. The state s_j at time $n+1$ — and the output code word — depends on the current state and an M-vector of source symbols. The mapping from state s_i to code word $\mathbf{c}_i$ is unique, which is not necessarily the case in the other direction. The elements p_{ij} of the $(K \times K)$ state transition matrix Π are the probabilities of the transition from state s_i to state s_j. We will also need a vector $\mathbf{p} = [p_0, \ldots, p_{K-1}]$ containing the steady-state probabilities p_i of states s_i. It can be obtained as the unit-norm eigen vector of Π corresponding to the eigen value equal to unity [2]. Complete statistics of the Markov chain are thus provided by Π. The autocorrelation function of the encoded sequence can be computed after Π, as well as all the possible code words, are defined.

First, let us construct a matrix $\mathbf{C}$ containing the K code words $\mathbf{c}_i$ corresponding to the K states s_i

$$\mathbf{C} = \begin{bmatrix} c_0(0) & \dots & c_0(N-1) \\ \vdots & \ddots & \vdots \\ c_{K-1}(0) & \dots & c_{K-1}(N-1) \end{bmatrix} \quad (1)$$

and define a column vector $\mathbf{c}(i) = [c_0(i), \dots, c_{K-1}(i)]^T$. The discrete autocorrelation function of the encoded sequence is then [1]

$$R_x(j) = \sum_k \sum_i \mathbf{c}^H(i)\mathbf{R}(k)\mathbf{c}(i+j-Nk) \quad (2)$$

where the range of i and k is defined by ($0 \geq i \leq N-1 \cap 0 \geq i+j-Nk \leq N-1$). The state correlation matrix is defined as

$$\mathbf{R}(k) = \begin{cases} \mathbf{D}\Pi^k & , k \geq 0 \\ (\Pi^H)^{-k}\mathbf{D} & , k < 0 \end{cases} \quad (3)$$

where $\mathbf{D}$ is $diag(p_0, \dots, p_{K-1})$.

3. LINEAR BLOCK CODES

In the previous section, we presented a generic method for computing the autocorrelation function of a channel encoded sequence. In this paper, we concentrate on linear block codes. Since we are using a Mealy machine, the Markov model for a linearly block encoded sequence is trivial. For an L-ary code, the number of states $K = L^M$. The transition probability from state s_i to any of the K states s_j — as well as every element in the $(K \times K)$-matrix Π — equals $1/K$:

$$\Pi = \begin{bmatrix} 1/K & \dots & 1/K \\ & \ddots & \\ 1/K & \dots & 1/K \end{bmatrix} \quad (4)$$

Also, the steady state probabilities $p_i = 1/K$, for all $i = 0, \dots, K-1$, so that

$$\mathbf{D} = diag(1/K, \dots, 1/K) \quad (5)$$

Consequently, $\mathbf{R}(k) = \mathbf{D}$ for $k = 0$ and $\mathbf{R}(k) = \mathbf{D}\Pi$ otherwise. By constructing the code word matrix (1), we can compute the autocorrelation sequence $R_x(j)$ from (2). The order of the row vectors in $\mathbf{C}$ is now arbitrary, because all the Markov probabilities in Π are equal.

3.1. Binary codes

A practical representation of a linear block code is a generator matrix. In this paper, we first employ the binary $M = 4$ and $N = 7$ block code of Fig. 1. The code word $\mathbf{c}_i$ is obtained from a block $\mathbf{a}_M$ of the information sequence as

$$\mathbf{c}_i = \mathbf{a}_4\mathbf{G} \quad (6)$$

where the generator matrix [9]

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \quad (7)$$

This is a systematic Hamming code, with $\mathbf{G}$ consisting of an identity matrix and a parity sequence generating part $\mathbf{G} = [\mathbf{I}|\mathbf{P}]$. The number of states and code words is now $K = 2^4$. Matrix (1) can be constructed by using all permutations of the source sequence in (6). The top plot of Fig. 2 shows the analytical $R_x(j)$ (2) of the encoded sequence $x(n)$. Clearly, $R_x(j) = \delta(j)$, which implies that algorithms using the whiteness assumption of the data should not suffer any performance degradation when the encoded data is used. The analytical result is verified by the corresponding sample estimated autocorrelation $\hat{R}_x(j) = 1/N_s \Sigma_{n=1}^{N_s}[x(n)x^*(n-j)]$, where $N_s = 2000$, in the same plot. It should be noted that the encoded sequence is cyclostationary and its second order statistics not completely determined by $R_x(j)$ [5].

Not all block codes result in perfectly white sequences. We demonstrate this by evaluating $R_x(j)$ for all the possible generator matrices of a systematic $M = 4, N = 7$ code that have unique and non-zero columns in $\mathbf{P}$. There are 4096 combinations in $\mathbf{P}$ and 2730 of them fulfill the conditions. The bottom plot of Fig. 2 shows the average analytical autocorrelation function of these sequences. The number of perfectly white sequences is 990.

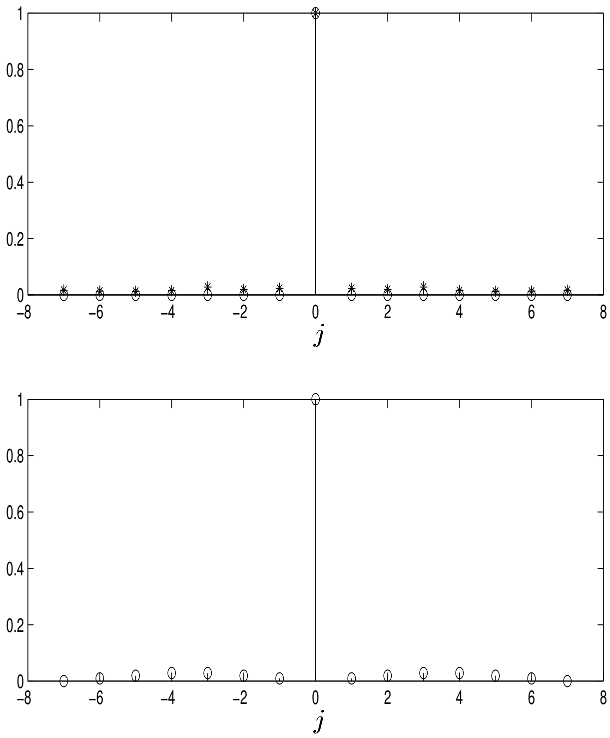

Figure 2: Autocorrelation of a linearly block encoded sequence, 'o' analytical, '*' sample estimated. Top plot: generator matrix (7). Bottom plot: average of 2730 $(7, 4)$-systematic codes.

3.2. L-ary Reed-Solomon codes

Some codes considered in the previous section may be non-practical. In order to study a more realistic example, we evaluate the autocorrelation properties of Reed-Solomon (RS) codes. This family of cyclic block codes has been applied in practice in e.g. space technology [10]. The lengths of the L-ary source and code words are related by $N = L - 1 = 2^k - 1, M = 1, \ldots, N - 1$.

We employ a $N = 7, M = 3$ 8-QAM RS-code (or equivalently a binary $N_b = 21, M_b = 9$ code). There are now a total of $K = 8^3$ code words and states. Constructing the matrices needed in (2) and evaluating the expression for $R_x(j)$ results in the analytical autocorrelation function shown in the top plot of 3. For a $N = 15, M = 5$ 16-QAM code, there are $K = 16^5$ distinct code words. Computations on the $(K \times K)$-matrix Π are now too expensive and we settle for presenting the sample estimated autocorrelation function $\tilde{R}_x(j)$ at the bottom plot of Fig. 3. Clearly, for the both RS-codes, the whiteness assumption holds.

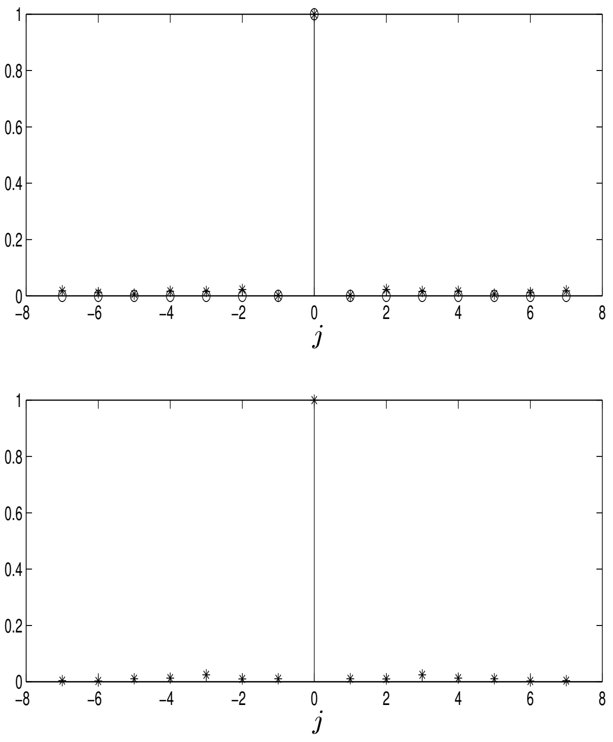

Figure 3: Autocorrelation function, 'o' analytical, '*' sample estimated. Top plot: $(7, 3)$ RS code. Bottom plot: $(15, 5)$ RS code.

4. BLIND EQUALIZER PERFORMANCE FOR WHITE AND ENCODED DATA

In this section, the performance of the prediction-error filter (PEF) based blind equalizer [8] is evaluated in simulation for unencoded and encoded channel inputs. The employed equalizer employs a Single-Input Multiple-Output (SIMO) model, i.e. fractional sampling. The equalizer is computed from the PEF coefficients $\mathbf{A} = [\mathbf{I} \quad -\mathbf{w}]$ and the first channel coefficient column vector $\mathbf{h}(0)$ as $\mathbf{g}(n) = \mathbf{h}^H(0)/[\mathbf{h}^H(0)\mathbf{h}(0)]\mathbf{A}$. The predictor coefficients are obtained from the Yule-Walker equations as $\mathbf{w} = \mathbf{r}\mathbf{R}^{-1}$, where $\mathbf{R}$ is the block-Toeplitz autocorrelation matrix and $\mathbf{r}$ is the lagged autocorrelation of the received sequence. See [11, 8] for details of the SIMO model and the equalizer algorithm.

The unencoded sequence is white 4-QAM with unit variance. The I- and Q-components of the sequence are independently encoded with the linear block code of (7). The additive noise is white and Gaussian. The SNR is defined as

$$\text{SNR} := \frac{\text{E}[\Sigma_{j=0}^{P-1} |y'_j(n)|^2]}{\text{E}[\Sigma_{k=0}^{P-1} |v_k(n)|^2]} \tag{8}$$

where $y'_j(n)$ is the signal component of $y_j(n)$. The employed communication channel (Fig. 4) was obtained from the web-site *http://spib.rice.edu/spib/microwave.html* and was further time-decimated to length 16 as explained in [3]. At each evaluated point, 5000 test symbols were used to compute the estimate of the MSE and the procedure was repeated 100 times. No error detection/correction has been performed.

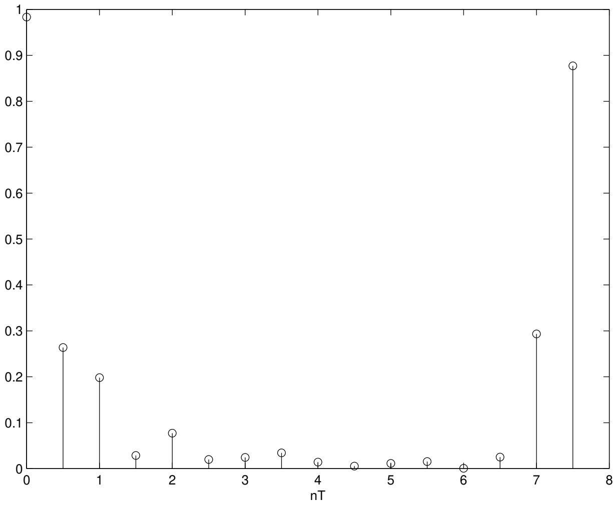

Figure 4: Magnitude of the channel impulse response.

Fig. 5 shows the MSE of the blind equalizer for the two input sequences at different SNRs and a sample size of 1000. No major difference in the performance can be noted between the white and the encoded sequence. The same observation can be made from Fig. 6, where the MSE is plotted for the SNR of 15 dB as a function of the sample size. Further confirmation is obtained from Fig. 7, where the SER is similarly plotted for the SNR of 10 dB.

5. CONCLUSIONS

The autocorrelation function of a linearly block encoded data sequence was studied. An analytical expression based on a Markov model was given. The analysis was performed for several block codes and the results were

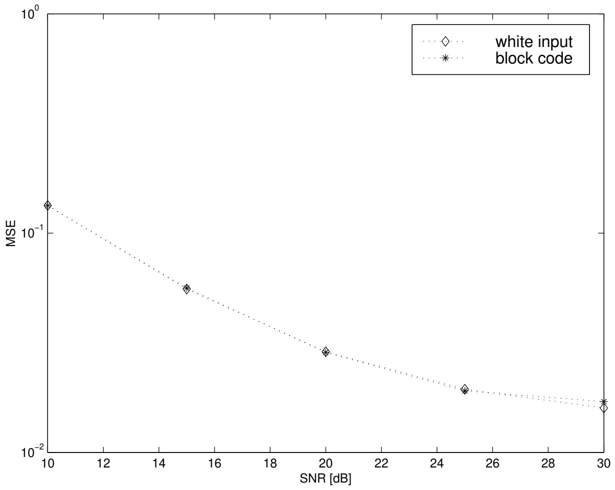

Figure 5: MSE of the blind equalizer for unencoded and linear block encoded sequences, sample size 1000.

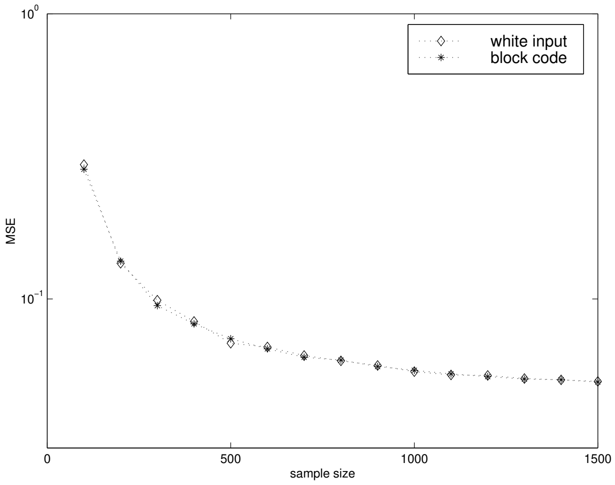

Figure 6: Convergence curves of the blind equalizer for unencoded and linear block encoded sequences, SNR 15 dB.

verified with Monte-Carlo simulations. Furthermore, a block encoded sequence was used in prediction-based blind equalization in comparison with a white sequence.

From the analytical and simulation results of this paper, it can be noted that blind equalizer algorithms that are derived assuming white input sequences are applicable to systems employing linear block encoded data. It will be interesting to perform similar analysis for other types of channel codes, e.g. convolutional codes.

6. REFERENCES

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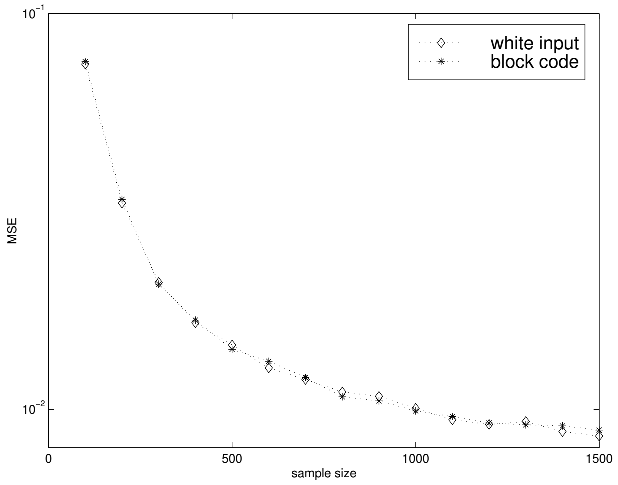

Figure 7: Convergence of the SER of the PEF-based blind equalizer with the SNR of 10 dB.

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