ERROR ANALYSIS OF A MULTI-STEP PREDICTION BASED BLIND EQUALIZER

Jukka Mannerkoski and Visa Koivunen*

Signal Processing Laboratory, Tampere University of Technology
P.O. Box 553, FIN-33101 Tampere, Finland
fax: +358-3-365 3857
email: {jhm, visa}@cs.tut.fi

Keywords – blind channel equalization, linear prediction, second-order cyclostationarity.

ABSTRACT

Blind equalization is used to compensate the inter-symbol interference caused by a communication channel without using any known training sequences. For many practical channel types, a suitable delay at the output of the equalizer allows for achieving a small estimation error. Blind equalizers may be implemented with linear prediction-error filters (PEF), but the delay cannot be controlled with one-step predictors. Consequently, multi-step prediction has been suggested as a solution to the problem. We consider the effects of additive noise at the output of the multi-step PEF. An analytical error bound for a PEF-based blind zero-forcing (ZF) equalizer in the presence of noise is derived. The obtained results are verified with simulations.

1. INTRODUCTION

A communication channel introduces inter-symbol interference (ISI) in the received sequence. Blind equalization attempts to remove the ISI without the knowledge of the transmitted sequence. If multiple samples per symbol are available at the receiver, blind equalizers can be derived from the second-order cyclostationary statistics of the received signal [2, 6]. Zero-forcing (ZF) and minimum mean-square error (MMSE) equalizers can be implemented with linear multichannel prediction-error filters (PEF) [4, 5]. However, one-step forward prediction can only result in zero-delay equalization and one-step backward prediction in maximum-delay equalization. For many practical channels, a smaller equalization error may be achieved by controlling the delay. The selected delay should reflect the distribution of energy in the channel impulse response.

Multi-step prediction has been suggested as a solution to the arbitrary-delay equalization problem [5, 3]. The approach is based on the assumption of zero noise, so that the multi-step PEF perfectly reduces the degree of the ISI in the received sequence. Consequently, the prediction error can be modeled as an output of a truncated channel with no additive noise. Based on this model, a simple blind equalizer employing two multi-step PEFs was introduced in [3]. When the PEF outputs are properly combined, the transmitted symbol can be reconstructed without error (ZF-equalization).

We study how the additive noise affects the interpretation of the PEF output as an output of the truncated channel. In the presence of additive noise, the PEF output includes a colored noise component. Moreover, the multi-step PEF is no longer able to reduce the degree of the ISI perfectly, but contains an interference component from the previous symbols at its output. The residual ISI and the colored noise introduce error in the system output. We derive an analytical expression for the error.

As a consequence of the noise induced modeling error, the blind equalizer can no longer reconstruct the transmitted symbols perfectly. A lower bound for the mean-square error (MSE) of the symbol estimate is derived. Simulations are used to verify the analytical results.

2. SYSTEM MODEL

Let $x(k)$ denote the symbol emitted by a random digital source at time kT, T being the symbol period. The channel is considered to be linear and time-invariant (LTI) during a time window sufficient to allow its estimation. The continuous time received signal is

$$y(t) = \sum_{k=-\infty}^{\infty} x(k)h(t-kT) + v(t) \tag{1}$$

where $h(t)$ is the complex baseband channel impulse response, $x(k)$ is the sequence of complex information symbols and $v(t)$ is additive noise.

When $P \geq 2$ samples of $y(t)$ per symbol interval are available at the receiver, the system can be modeled as consisting of P subchannels. The ith subchannel response is defined as $h_i(n) \triangleq h(t_0 + iT/P + nT)$, $n = 0, \ldots, L_h - 1$, where L_h is the subchannel length. Its output sequence $y_i(n) \triangleq y(t_0 + iT/P + nT)$ is

$$y_i(n) = \sum_{k=0}^{L_h-1} x(n-k)h_i(k) + v_i(n) \tag{2}$$

where $i = 0, \ldots, P-1$, $-\infty < n < \infty$, $v_i(n)$ are samples of $v(t)$ corresponding to $y_i(n)$ and the symbol period is normalized to $T = 1$. In each symbol interval, a vector $\mathbf{y}(n) = [y_0(n), \ldots, y_{P-1}(n)]^T$ of length P is received, $[\ ]^T$ denoting transposition. The channel impulse response can also be represented in vector form as

* This work was funded by the Academy of Finland

$\mathbf{h}(n) = [h_0(n), \ldots, h_{P-1}(n)]^T$ and the noise as $\mathbf{v}(n) = [v_0(n), \ldots, v_{P-1}(n)]^T$. We consider a finite symbol sequence of length M and define a matrix $\mathcal{H}_M$ with dimension $MP \times (L_h + M - 1)$ consisting of the channel coefficient vectors,

$$\mathcal{H}_M = \begin{bmatrix} \mathbf{h}(0) & \ldots & \mathbf{h}(L_h-1) & & \mathbf{0} \\ & \ddots & & \ddots & \\ \mathbf{0} & & \mathbf{h}(0) & \ldots & \mathbf{h}(L_h-1) \end{bmatrix} \tag{3}$$

Stacking M received vector samples into an $(MP \times 1)$-vector $\mathbf{y}_M(n) = [\mathbf{y}^T(n), \ldots, \mathbf{y}^T(n-M+1)]^T$, we can write a matrix equation for the system as

$$\mathbf{y}_M(n) = \mathcal{H}_M \mathbf{x}_{L_h+M-1}(n) + \mathbf{v}_M(n) \tag{4}$$

where the input vector is $\mathbf{x}_{L_h+M-1}(n) = [x(n), \ldots, x(n-L_h-M+2)]^T$ [7]. At this stage, we state the assumptions used throughout the paper:

A1. The input sequence $x(t)$ is zero-mean and white with unit variance.
A2. The additive noise $v(t)$ is zero-mean and white with variance σ_v^2.
A3. The sequences $x(t)$ and $v(t)$ are uncorrelated.
A4. The matrix $\mathcal{H}_M$ has full column rank, i.e. the subchannels $h_i(n)$ have no common zeros.
A5. The dimensions of $\mathcal{H}_M$ obey $MP \geq L_h + M - 1$.

3. MULTI-STEP FORWARD PREDICTION-ERROR FILTERING

A k-step forward predictor of order M produces an estimate $\hat{\mathbf{y}}(n)$ of the received symbol $\mathbf{y}(n)$ based on the M previous symbols $\mathbf{y}_M(n-k)$. The filter coefficients $\mathbf{w}_{M,k}$ are obtained from Wiener-Hopf equations. A corresponding k-step forward prediction error filter (PEF) of order M produces the error $\mathbf{y}(n) - \hat{\mathbf{y}}(n)$ as its output. The PEF coefficients are represented in matrix form as $\mathbf{A}_{M,k} = [\mathbf{I}_P \; \mathbf{0}_{P\times P(k-1)} \quad -\mathbf{w}_{M,k}]$. The k-step prediction error is then

$$\mathbf{f}_{M,k}(n) = \mathbf{A}_{M,k}\mathbf{y}_{M+k}(n) \tag{5}$$

The PEF output $\mathbf{f}_{M,k}(n)$ can be given a meaningful interpretation in the context of channel equalization as the output of a truncated channel $\bar{\mathbf{h}}(n) \triangleq \mathbf{h}(n), n = 0, \ldots, k-1$.

Theorem 1 *The output $\mathbf{f}_{M,k}(n)$ of the k-step PEF is given by*

$$\mathbf{f}_{M,k}(n) = \sum_{j=0}^{k-1} \mathbf{h}(j)x(n-j) + \xi_{M,k}(n) \tag{6}$$

where $\xi_{M,k}(n) = \sigma_v^2 \mathbf{w}_{M,k}\mathcal{H}_M^{\dagger H}\mathbf{x}_{L_h+M-1}(n-k) + \mathbf{A}_{M,k}\mathbf{v}_{M+k}(n)$, $\mathcal{H}_M^{\dagger}$ is the pseudo-inverse of $\mathcal{H}_M$ and $MP \geq L_h + M - 1$.

The proof is in the Appendix. In the noiseless case, the number of transmitted symbols affecting any received symbol (degree of ISI) is reduced from L_h to k. The additive noise introduces an unwanted component $\xi_{M,k}(n)$ comprising of residual ISI and colored noise at the output of the partial channel. Fig. 1 summarizes the ideas presented above.

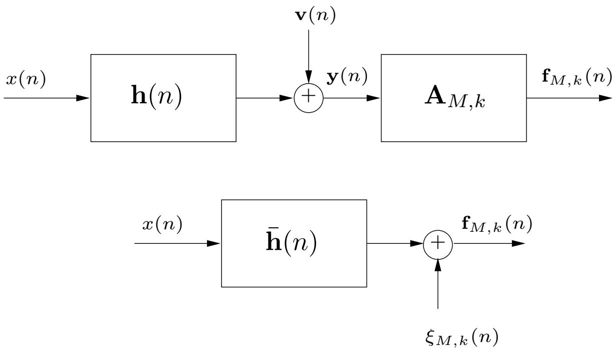

Figure 1: The output of a k-step PEF as an output of a truncated channel.

The covariance matrix $J_\xi = \mathrm{E}[\xi_{M,k}(n)\xi_{M,k}^H(n)]$ of the error in (6) is expanded in Theorem 2 ($[\;]^H$ denotes Hermitian transposition).

Theorem 2 *The covariance matrix of the residual ISI plus noise at the $\bar{\mathbf{h}}(n)$ output is*

$$J_\xi = \sigma_v^4 \mathbf{w}_{M,k}\mathcal{H}_M^{\dagger H}\mathcal{H}_M^{\dagger}\mathbf{w}_{M,k}^H + \sigma_v^2 \mathbf{A}_{M,k}\mathbf{A}_{M,k}^H \tag{7}$$

The proof is obvious from (6) using assumptions A1 – A3.

4. BLIND EQUALIZATION BASED ON MULTI-STEP PREDICTION

The blind equalization method considered here is based on the outputs of k- and $(k+1)$-step PEFs [3]. A k-delay ZF-equalizer can be obtained after acknowledging from (6) that in the absence of noise

$$\mathbf{F}_{M,k}(n) \triangleq \mathbf{f}_{M,k+1}(n) - \mathbf{f}_{M,k}(n) = \mathbf{h}(k)x(n-k) \tag{8}$$

The transmitted symbol can then be extracted as $x(n-k) = \nu(k)\mathbf{F}_{M,k}(n)$, where $\nu(k) = \mathbf{h}^H(k)/[\mathbf{h}^H(k)\mathbf{h}(k)]$. The channel coefficient vector $\mathbf{h}(k)$ can be estimated from the correlation matrix of $\mathbf{F}_{M,k}(n)$ [1]. The additive noise introduces an error $\nu(k)[\xi_{M,k+1}(n) - \xi_{M,k}(n)]$ in the symbol estimate (c.f. (6) and (8)). The MSE of the symbol estimate is thus $J_{\min} = \nu(k)\mathrm{E}\{[\xi_{M,k+1}(n) - \xi_{M,k}(n)]\,[\xi_{M,k+1}(n) - \xi_{M,k}(n)]^H\}\nu^H(k)$ which is expanded below.

Theorem 3 *A lower bound for the MSE of the multi-step prediction based blind equalizer output is*

$$J_{min} = \nu(k)[J_\gamma + J_v]\nu(k)^H \tag{9}$$

where

$$\begin{aligned} J_\gamma &\triangleq \gamma_{M,k+1}\gamma_{M,k+1}^H + \gamma_{M,k}\gamma_{M,k}^H \\ &\quad -\gamma_{M,k+1}\mathbf{D}_1\gamma_{M,k}^H - \gamma_{M,k}\mathbf{D}_1^H\gamma_{M,k+1}^H \\ J_v &\triangleq \sigma_v^2[\mathbf{A}_{M,k+1}\mathbf{A}_{M,k+1}^H + \mathbf{A}_{M,k}\mathbf{A}_{M,k}^H \\ &\quad -\mathbf{A}_{M,k+1}\mathbf{I}_{left}\mathbf{A}_{M,k}^H - \mathbf{A}_{M,k}\mathbf{I}_{left}^H\mathbf{A}_{M,k+1}^H] \end{aligned} \tag{10}$$

where the $P \times (L_h + M - 1)$*-matrix* $\gamma_{M,k} \triangleq \sigma_v^2 \mathbf{w}_{M,k}\mathcal{H}_M^{\dagger H}$, $\mathbf{D}_1$ *is an* $(L_h + M - 1)$*-square matrix with ones on its first diagonal above the main diagonal and zeros elsewhere and* $\mathbf{I}_{left} = [\mathbf{I}_{P(M+k)}\ \mathbf{0}_{P\times P(M+k)}]$.

The proof follows from (6) after recognizing assumptions A1 – A3. The component J_γ is due to the residual ISI and J_v to the colored noise at the output of the multi-step PEF.

5. ANALYTICAL AND SIMULATED RESULTS

In this section, we evaluate the analytical expression for the residual ISI plus noise at the k-step PEF output. We also compute a lower bound for the blind equalizer MSE and verify the result in simulation. The magnitude of the employed channel is shown in Fig. 2. It is an empirically measured microwave channel obtained from the web-site *http://spib.rice.edu/spib/microwave.html*. We use $k = 6$ for this particular channel. The source sequence is white 4-QAM and the additive noise is white and Gaussian. For the simulations, a sample size of 25 000 is used to estimate the correlation matrices. An independent test set of 5000 symbols is used at each evaluated point and the results are averaged over 100 Monte-Carlo runs.

The output of a k-step forward PEF was shown in (6) to be an output of a partial channel $\bar{\mathbf{h}}(n)$. The trace of the covariance matrix of the output error, as given in (7), is plotted as a function of the SNR in Fig. 3. The overall error consists of the residual ISI component and the colored noise, which are also plotted. At low SNRs, the majority of the error is due to the residual ISI component. It is attenuated rapidly as the noise power decreases as a consequence of its dependence on σ_v^4 (c.f (7)). Next we evaluate the performance of the blind equalizer. The analytical and simulated MSE of the equalizer output are shown in Fig. 4 as a function of the SNR. The simulated results are on par with the analytical ones at low SNRs. At high SNRs the finite sample size causes an error floor in the simulation results. The MSE is plotted for different values of k in Fig. 5. The best performance is obtained when k is large, which is a reasonable result, because the energy of the channel is concentrated in the end of the impulse response.

6. CONCLUSIONS

The rationale behind using multi-step prediction in blind equalization lies in the need to control the delay of the equalizer. The performance of a previously suggested blind equalizer — and the limitations of a model it is based on — were analyzed. The assumption of zero noise used in deriving the equalizer is not valid for real communication channels, which introduces uncontrolled error in the equalizer output. An analytical lower bound for the error was derived and evaluated for an empirically measured microwave channel. Simulations were used to verify the results.

7. APPENDIX

Proof of Theorem 1: Starting from the orthogonality condition

$$\mathrm{E}\{\mathbf{y}_M(n-k)\mathbf{f}_{M,k}^H(n)\} = \mathbf{0}_{MP\times P} \tag{11}$$

we extend the proof provided in [3] by considering the additive noise as well. Using (5) and (4), we get

$$\begin{aligned} &\mathrm{E}\{[\mathcal{H}_M\mathbf{x}_{L_h+M-1}(n-k) + \mathbf{v}_M(n-k)] \\ &\quad [\mathcal{H}_{M+k}\mathbf{x}_{L_h+M+k-1}(n) + \mathbf{v}_{M+k}(n)]^H\}\mathbf{A}_{M,k}^H \\ &\quad = \mathbf{0}_{MP\times P} \end{aligned} \tag{12}$$

An $i \times j$ matrix of zeros is denoted $\mathbf{0}_{i\times j}$ and an $i \times i$ unity matrix $\mathbf{I}_i$. The input sequence and the additive noise are assumed uncorrelated, zero-mean and white, so that evaluating (12) and premultiplying with $\mathcal{H}_M^\dagger$ results in

$$\begin{aligned} &[\mathbf{0}_{(L_h+M-1)\times k}\ \mathbf{I}_{L_h+M-1}]\mathcal{H}_{M+k}^H\mathbf{A}_{M,k}^H \\ &\quad - \sigma_v^2\mathcal{H}_M^\dagger\mathbf{w}_{M,k}^H = \mathbf{0}_{(L_h+M-1)\times P} \end{aligned} \tag{13}$$

with $\mathcal{H}_M^\dagger\mathcal{H}_M = \mathbf{I}$, when $MP \geq L_h + M - 1$, i.e. $\mathcal{H}_M$ has more rows than columns. Let us denote the first k rows of $\mathcal{H}_{M+k}^H$ with $\mathcal{H}^k$ (we assume that $k \leq L_h - 1$) and the remaining rows with $\mathcal{H}^f$. The effect of the matrix multiplier $[\mathbf{0}_{(L_h+M-1)\times k}\ \mathbf{I}_{L_h+M-1}]$ is to remove $\mathcal{H}^k$ from $\mathcal{H}_{M+k}^H$. Consequently, we can write (13) as

$$\mathcal{H}^f\mathbf{A}_{M,k}^H = \sigma_v^2\mathcal{H}_M^\dagger\mathbf{w}_{M,k}^H \tag{14}$$

Realizing that $\mathbf{A}_{M,k} = [\mathbf{I}_P\ \mathbf{0}_{P\times P(k-1)}\ \ -\mathbf{w}_{M,k}]$, inserting $\mathcal{H}^k$ on top of $\mathcal{H}^f$ in (14) results in

$$\mathcal{H}_{M+k}^H\mathbf{A}_{M,k}^H = \begin{bmatrix} \mathbf{h}^H(0) \\ \vdots \\ \mathbf{h}^H(k-1) \\ \sigma_v^2\mathcal{H}_M^\dagger\mathbf{w}_{M,k}^H \end{bmatrix} \tag{15}$$

Transposing and postmultiplying by $\mathbf{x}_{L_H+M+k-1}(n)$ and adding a noise term on both sides, we get

$$\begin{aligned} \mathbf{f}_{M,k}(n) &= \sum_{j=0}^{k-1}\mathbf{h}(j)x(n-j) \\ &+\sigma_v^2\mathbf{w}_{M,k}\mathcal{H}_M^{\dagger H}\mathbf{x}_{L_H+M-1}(n-k) + \mathbf{A}_{M,k}\mathbf{v}_{M+k}(n) \end{aligned} \tag{16}$$

which is the desired result. □

8. REFERENCES

[1] K. Abed-Meraim, P. Duhamel, D. Gesbert, P. Loubaton, S. Mayrargue, E. Moulines, and D. Slock. Prediction error methods for time-domain blind identification of multichannel fir filters. In *Proc. of Int'l Conf. on Acoust. Speech and Signal Proc.*, pages 1968–1971, Detroit, MI, 1995.

[2] W. A. Gardner. A new method of channel identification. *IEEE Trans. Commun.*, 39(6):813–817, June 1991.

[3] D. Gesbert and P. Duhamel. Robust blind channel identification and equalization based on multi-step predictors. In *Proc. of Int'l Conf. on Acoust. Speech and Signal Proc.*, pages 3621–3624, 1997.

[4] D. T. Slock. Blind fractionally-spaced equalization, perfect reconstruction filter banks and multichannel linear prediction. In *Proc. of Int'l Conf. on Acoust. Speech and Signal Proc.*, pages IV585–IV588, Adelaide, Australia, April 1994.

[5] D. T. Slock and C. B. Papadias. Further results on blind identification and equalization of multiple fir channels. In *Proc. of Int'l Conf. on Acoust. Speech and Signal Proc.*, pages 1964–1967, Detroit, MI, May 1995.

[6] L. Tong, G. Xu, and T. Kailath. Blind identification and equalization based on second-order statistics: A time-domain approach. *IEEE Trans. Info. Theory*, 40(2):340–349, March 1994.

[7] L. Tong, G. Xu, and T. Kailath. Blind identification based on second-order statistics: A frequency-domain approach. *IEEE Trans. Info. Theory*, 41(1):329–334, January 1995.

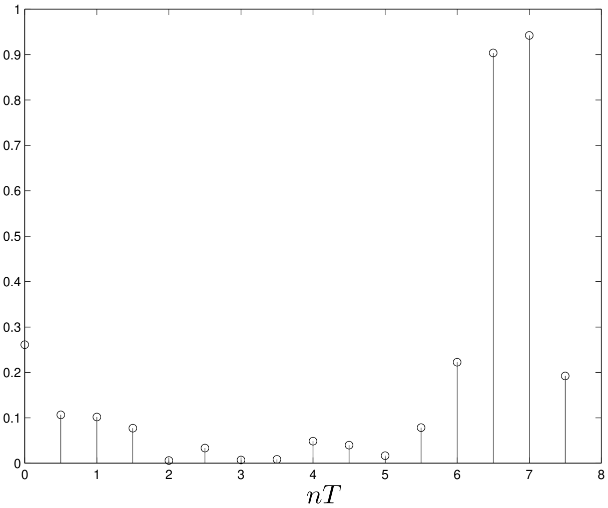

Figure 2: Magnitude of the channel impulse response.

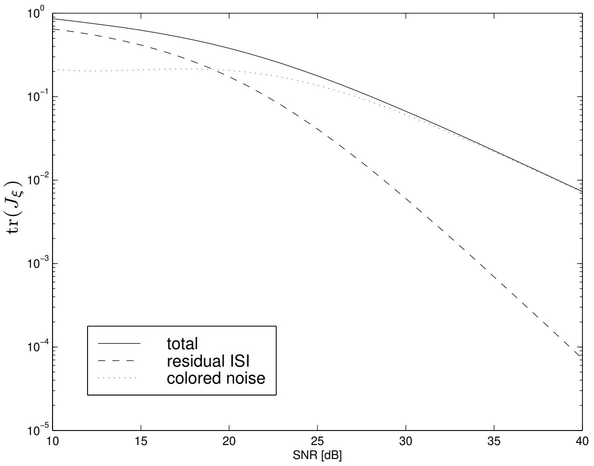

Figure 3: Error at the output of the partial channel.

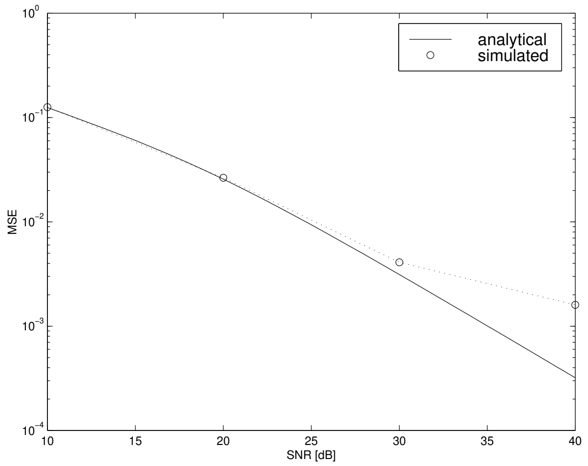

Figure 4: MSE of the blind equalizer output.

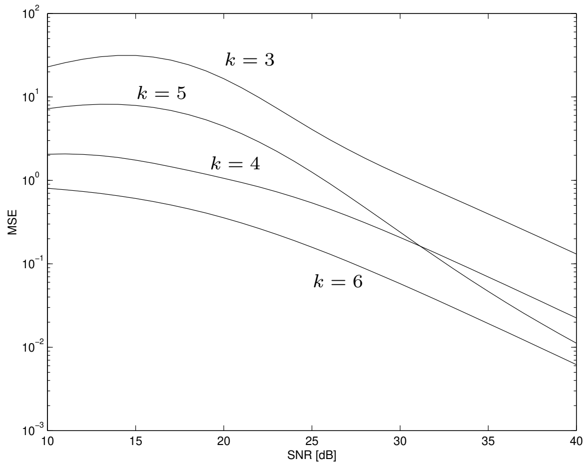

Figure 5: Effect of k on the MSE of the blind equalizer output.