Blind Equalization using Least-Squares Lattice Prediction

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Abstract

Second-order statistics of the received signal can be used to equalize a communication channel without knowledge of the transmitted sequence. Blind zero-forcing (ZF) and minimum mean-square error (MMSE) equalization can be achieved with linear prediction-error filtering. The equivalence with the equalizers derived by Giannakis and Halford [1] is shown and adaptive predictors that result in a lattice filtering structure are applied. The required channel coefficient vector is obtained with adaptive eigen-pair tracking. Either forward or backward prediction errors can be used. The performance of the blind equalizer is examined by simulations. The MMSE of the optimum FSE is approached and the algorithm exhibits robustness to channels with common subchannel zeros.

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1 Introduction

Multipath fading is encountered in digital wireless communications. Equalization or channel estimation combined with maximum-likelihood sequence estimation is used to compensate the resulting intersymbol interference (ISI) [2]. Equalization is often preferred due to its simplicity of implementation. The usual approach is to transmit a training signal and to update the equalizer coefficients according to the resulting error signal. Training has to occur periodically, because the channel is time-variant. Drawbacks of the method — wasted bandwidth, possible deep fades during training and difficulty of implementation in broadcast systems — have motivated the development of blind algorithms that adapt to the channel without training (cf. [3], [4]).

The early blind algorithms [5], [6] and [7] use higher order statistics (HOS) of the received signal in an implicit way. The non-linear function used at the output of the equalizer results in complicated analysis and the convergence properties of these algorithms have been often misunderstood [8]. The fractionally spaced constant modulus algorithm (FS-CMA) was shown in [9] to guarantee global convergence with finite length equalizers. The FS-CMA is attractive for applications, if ease of implementation is critical. Explicit use of HOS in [10], [11] and [12] provides a good basis for mathematical models. The resulting algorithms are not applicable to high-rate communication systems, because they require large sample sets and have high computational complexity. Gardner [13] established the possibility of equalization/identification of a non-minimum phase channel from the second-order statistics (SOS) of the received signal. Several samples per symbol must be available and certain restrictions on the channel must hold. Consequent SOS-based algorithms include [14], [15], [16], [17], [18] and [1]. Most SOS-algorithms adapt their parameters after a block of data has been received, recent exceptions being [1] and [19]. They often provide a channel estimate and further processing is needed to obtain the symbol estimates. Most algorithms are sensitive to channel order estimation.

The present work develops a fast converging blind equalizer able to adapt to the channel variations after each received symbol. The approach is based on linear prediction methods [15], [20] and [21]. The resulting algorithm has a natural recursive structure and it completely avoids inversion of large matrices. An adaptive tracking algorithm is used to obtain the required channel coefficient vector. Zero-forcing (ZF) and minimum mean-square error (MMSE) equalizers can be obtained directly.

In section 2 we present the model of an oversampled communication system. Section 3 describes how blind equalization can be achieved by forward or backward linear prediction. An adaptive least-squares lattice (LSL) algorithm is used as the linear predictor and eigen-pair tracking to obtain the unknown channel coefficient vector. We call the resulting structure the lattice blind equalizer (LBE). Simulation results are presented in section 4.

2 System model

2.1 The multichannel communication system

A model of a baseband communication system is presented in Fig. 1. Let $x(k)$ denote the symbol emitted by a random digital source at time kT, T being the symbol period. The channel is considered linear and time-invariant (LTI) during a time window sufficient to allow its estimation. The continuous time received signal is

$$y(t) = \sum_{k=-\infty}^{\infty} x(k)h(t-kT) + v(t) \tag{1}$$

where $h(t)$ is the complex baseband channel impulse response, $x(k)$ the sequence of complex information symbols and $v(t)$ additive noise. The response $h(t)$ includes the transmit filter, the transmission medium and possible receive filter and is assumed to have a finite duration of $L_h T$, where L_h is an integer [2].

When several samples per symbol interval ($P > 1$ in Fig. 1) are taken of $y(t)$, the over-sampled single-input single-output system model of Fig. 1 can be represented as a single-input multiple-output (SIMO) model as in Fig. 2. The ith subchannel response is defined as $h_i(n) \triangleq h(t_0 + iT/P + nT),\ n = 0, \ldots, L_h - 1$. Its output sequence $y_i(n) \triangleq y(t_0 + iT/P + nT)$ is

$$y_i(n) = \sum_{k=0}^{L_h-1} x(n-k)h_i(k) + v_i(n) \tag{2}$$

where $i = 0, \ldots, P-1$, $-\infty < n < \infty$, $v_i(n)$ are samples of $v(t)$ corresponding to $y_i(n)$ and the symbol period is normalized to $T = 1$. The maximum length of $h_i(n)$ is L_h. An equivalent model results from implementing P receivers and taking one sample per symbol at each of them.

The received sequence $y(n/P)$ is cyclostationary (CS), which enables blind channel identification or equalization from the second-order statistics of the received sequence, if the $h_i(n)$ have no zero in common [22]. Blind identification/equalization can only be performed up to a multiplicative constant $e^{j\theta}$, so that for zero-forcing equalization, the symbol estimate is [17].

$$\hat{x}(n) = x(n)e^{j\theta} \tag{3}$$

This phase ambiguity is generally remedied with differential encoding.

At this stage, we adopt the following assumptions:

A1. The subchannels $h_i(n)$ do not have any common zero.

A2. The input sequence $x(n)$ is white with unit variance σ_x^2, i.e. $\mathrm{E}[x(k)x^*(l)] = \delta(k-l)$.

A3. The additive noise is white with variance σ_v^2, i.e. $\mathrm{E}[v(k)v^*(l)] = \sigma_v^2\delta(k-l)$.

In each symbol interval, a vector $\mathbf{y}(n) = [y_0(n), \ldots, y_{P-1}(n)]^T$ of length P is received, $()^T$ denoting transposition (Hermitian transposition is denoted $()^H$). The channel impulse response can also be represented in vector form as $\mathbf{h}(n) = [h_0(n), \ldots, h_{P-1}(n)]^T$ and the noise as $\mathbf{v}(n) = [v_0(n), \ldots, v_{P-1}(n)]^T$. The cyclostationary process $y(n/P)$ is thus transformed into a wide-sense stationary vector process.

We consider a finite symbol sequence of length M and define a matrix $\mathcal{H}_M$ with dimension $MP \times (L_h + M - 1)$ consisting of the channel coefficient vectors,

$$\mathcal{H}_M = \begin{bmatrix} \mathbf{h}(0) & \ldots & \mathbf{h}(L_h-1) & & \mathbf{0} \\ & \ddots & & \ddots & \\ \mathbf{0} & & \mathbf{h}(0) & \ldots & \mathbf{h}(L_h-1) \end{bmatrix} \tag{4}$$

Stacking M received vector samples into an $(MP \times 1)$-vector $\mathbf{y}_M(n) = [\mathbf{y}^T(n), \ldots, \mathbf{y}^T(n-M+1)]^T$, we can write a matrix equation as

$$\mathbf{y}_M(n) = \mathcal{H}_M \mathbf{x}_M(n) + \mathbf{v}_M(n) \tag{5}$$

where the input vector is $\mathbf{x}_M(n) = [x(n), \ldots, x(n - L_h - M + 2)]^T$ [17]. The block-Toeplitz correlation matrix $\mathbf{R}_M$ of $\mathbf{y}_M$ is

$$\mathbf{R}_M = \begin{bmatrix} \mathbf{r}(0) & \mathbf{r}(1) & \ldots & \mathbf{r}(M-1) \\ \mathbf{r}^H(1) & \mathbf{r}(0) & \ldots & \mathbf{r}(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{r}^H(M-1) & \mathbf{r}^H(M-2) & \ldots & \mathbf{r}(0) \end{bmatrix} \tag{6}$$

where $\mathbf{r}(i-j) = \mathrm{E}[\mathbf{y}(i)\mathbf{y}^H(j)]$.

2.2 Fractionally spaced equalization

In the receiver, ZF- or MMSE-equalizers can be used to extract the transmitted symbol estimate. The ith subchannel $h_i(n)$ is equalized by the filter $g_i(n)$, as shown in Fig. 3. The equalizer impulse response is in vector form,

$$\mathbf{g}(n) = [g_0(n), g_1(n), \ldots, g_{P-1}(n)]^T \tag{7}$$

where $n = 0, \ldots, L_g - 1$ and L_g is the length of the longest branch of the equalizer. If oversampling is achieved with several symbol rate receivers, this structure follows naturally. If one oversampling receiver is used, the outputs of the different branches of the filter bank must be properly delayed before the summer.

Equalization introduces a delay of d symbols relative to the transmitted sequence. Setting $L_g = M + 1$, a d-delay equalizer vector of length $(M + 1)P$ is constructed as

$$\mathbf{g}_d = [\mathbf{g}^T(0), \ldots, \mathbf{g}^T(M)] \tag{8}$$

and the symbol estimate is obtained from

$$\hat{x}(n - d) = \mathbf{g}_d \mathbf{y}_{M+1} \tag{9}$$

The optimum receiver in a linear modulation system consists of a filter matched to the actual channel, a symbol-rate sampler and a transversal equalizer [23]. The signal components of the properly frequency shifted versions of $y(t)$ are correlated, whereas the corresponding noise/interference components are not. By intentional aliasing, the periodic sampler exploits this frequency diversity that is implicit in the cyclostationary data signal, when excess bandwidth is available [24]. The matched filter preceding the sampler aligns the phases of the frequency shifted components, so that they add constructively in the aliased spectrum. The optimum receiver can be implemented by a fractionally spaced equalizer (FSE) that uses a tap spacing shorter than the symbol period. The Nyquist rate for the received signal can thus be achieved and aliasing of the equalizer input band edges avoided. This results in robustness to variations in the sampler timing phase. The matched filter can also be included in the equalizer and only one filter is needed. The multichannel equalizer shown in Fig. 3 is in effect an FSE.

A d-delay zero-forcing equalizer is derived ignoring additive noise. It enforces the condition $\hat{x}(n - d) = x(n)$. For finite SIMO channels, ZF-equalizers of finite length can be found if assumption A1 holds and the equalizer length $L_g \geq L_h - 1$ [15], [1]. ZF-equalizers do not perform optimally in the presence of noise. An MMSE-equalizer minimizes the cost function

$$J_{\mathrm{MSE}}(\mathbf{g}_d) = \mathrm{E}[|x(n) - \hat{x}(n)|^2] \tag{10}$$

and thus accounts for the noise. An MMSE-equalizer that has full knowledge of the channel can be obtained as in [25] and [26].

3 An adaptive lattice predictor as a blind equalizer

3.1 Linear prediction and blind equalization

Linear one-step forward prediction uses the samples $\mathbf{y}_M(n-1)$ to estimate the sample $\mathbf{y}(n)$ [27]. The forward prediction error of order M is defined as

$$\mathbf{f}_M(n) = \mathbf{A}_M \mathbf{y}_{M+1}(n) \tag{11}$$

where $\mathbf{A}_M = [\mathbf{I}, \mathbf{A}_{M,1}, \ldots, \mathbf{A}_{M,M}]$ and the $\mathbf{A}_{M,k}$ are $P \times P$ coefficient matrices of a forward prediction-error filter (PEF) of order M.

Linear one-step backward prediction uses the samples $\mathbf{y}_M(n)$ of a time series to estimate the previous sample $\mathbf{y}(n-M)$. The backward prediction error of order M is

$$\mathbf{b}_M(n) = \mathbf{B}_M \mathbf{y}_{M+1}(n) \tag{12}$$

where $\mathbf{B}_M = [\mathbf{B}_{M,0}, \ldots, \mathbf{B}_{M,M-1}, \mathbf{I}]$ consists of the $P \times P$ backward PEF coefficient matrices of order M.

The multichannel prediction-error filter of order M minimizes the trace of the covariance matrix of the prediction error [28]. The forward and backward PEF coefficient matrices can be obtained from the matrix equation [29]

$$\begin{aligned} &\begin{bmatrix} \mathbf{I} & \mathbf{A}_{M,1} & \ldots & \mathbf{A}_{M,M} \\ \mathbf{B}_{M,0} & \mathbf{B}_{M,1} & \ldots & \mathbf{I} \end{bmatrix} \mathbf{R}_{M+1} \\ &= \begin{bmatrix} \mathbf{P}_{f,M} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & \mathbf{P}_{b,M} \end{bmatrix} \end{aligned} \tag{13}$$

where $\mathbf{P}_{f,M}$ is the covariance matrix of $\mathbf{f}_M$ and $\mathbf{P}_{b,M}$ the covariance matrix of $\mathbf{b}_M$.

An MMSE-optimized prediction error filter can be used as a blind ZF- or MMSE-equalizer as shown for the forward case in [15] and [20]. One-step forward prediction results in zero-delay and one-step backward prediction in $(L_h + M - 1)$-delay equalization. Let us first consider MMSE-equalization.

Theorem 1 *A zero-delay MMSE-equalizer* $\mathbf{g}_0^{MMSE}$ *can be obtained as [20]*

$$\mathbf{g}_0^{MMSE} = \sigma_x^2 \mathbf{h}^H(0)\mathbf{P}_{f,M}^{-1}\mathbf{A}_M \tag{14}$$

where σ_x^2 *is the variance of the transmitted symbol. The* $(L_h + M - 1)$*-delay MMSE-equalizer is*

$$\mathbf{g}_{L_h+M-1}^{MMSE} = \sigma_x^2 \mathbf{h}^H(L_h - 1)\mathbf{P}_{b,M}^{-1}\mathbf{B}_M \tag{15}$$

The prediction-based MMSE-equalizers (14) *and* (15) *are equivalent to the correlation matrix based MMSE-equalizers of [1].*

The proof is provided in Appendix A.1.

Zero-forcing equalizers can also be obtained from the linear prediction-error problem. When noise power $\sigma_v^2 = 0$, the MMSE- and ZF-equalizer become equivalent, $\mathbf{g}_0^{\mathrm{ZF}} = \mathbf{g}_0^{\mathrm{MMSE}}$ and $\mathbf{g}_{L_h+M-1}^{\mathrm{ZF}} = \mathbf{g}_{L_h+M-1}^{\mathrm{MMSE}}$.

Theorem 2 *The zero-delay ZF-equalizer* $\mathbf{g}_0^{ZF}$ *can be obtained as [21]*

$$\mathbf{g}_0^{ZF} = \frac{\mathbf{h}^H(0)}{\|\mathbf{h}(0)\|^2}\mathbf{A}_M \tag{16}$$

where $\|\mathbf{h}(0)\|$ *is the Euclidean norm of* $\mathbf{h}(0)$*. The* $(L_h + M - 1)$*-delay ZF-equalizer is*

$$\mathbf{g}_{L_h+M-1}^{ZF} = \frac{\mathbf{h}^H(L_h - 1)}{\|\mathbf{h}(L_h - 1)\|^2}\mathbf{B}_M \tag{17}$$

The prediction-based ZF-equalizers (16) *and* (17) *are equivalent to the correlation matrix based ZF-equalizers of [1].*

The proof is again provided in Appendix A.1.

Theorem 1 states the equivalence of the PEF-based MMSE-equalizers (14) and (15) and the correlation matrix approach of [1]. In practice, sample estimates of the prediction error and its covariance matrix are used in (14) and (15) and in [1] the correlation matrix is estimated directly.

We can conclude from theorems 1 and 2 that the solution to the multichannel prediction-error problem provides equalizers, if the channel vector $\mathbf{h}(0)$ or $\mathbf{h}(L_h - 1)$ is known. It can be obtained from the prediction-error covariance matrix as will be shown. An MMSE-equalizer needs knowledge of σ_x^2, which is assumed to be known a priori and $\mathbf{P}_{f,M}$, which is available from the predictor computations. Knowledge of the channel length L_h is not required.

One-step prediction can only produce zero-delay or $(L_h + M - 1)$-delay equalizers. The need for an arbitrary d-delay equalizer in real communication systems was clearly demonstrated, for example, in [1]. A d-delay ZF-equalizer can be constructed by cascading a multi-step forward PEF with a one-step backward PEF, as shown in [20].

3.2 The adaptive least-squares-lattice predictor

Computation of the forward and backward PEFs from (13) requires sample estimation of the correlation matrix $\mathbf{R}_{M+1}$. Consequently, the equalizer coefficients can be modified only after a block of data has been received, assuming channel quasi-stationarity during the block.

The least-squares lattice (LSL) algorithm, described in Appendix A.2, obtains the forward and backward prediction errors adaptively. It involves both order-update and time-update recursions to adapt the reflection coefficients $\kappa_{f,m}(n)$ and $\kappa_{b,m}(n)$ of a lattice filter, as shown in Fig. 4. The prediction errors of order $m = 1, \ldots, M$ are computed with the aid of the computations pertaining to order $m-1$. The signals propagating in the filter of Fig. 4 are column vectors of length P and the summations, multiplications and storage are multidimensional operations.

The errors at different stages of a lattice filter are orthogonal, so that the prediction order can be increased (decreased) by simply adding (removing) stages to (from) the lattice [28]. This is attractive in high-rate communication systems, because an estimate is obtained whether or not the final order of the recursion has been reached when the allocated time slot expires. Lattice filters produce forward and backward prediction-errors simultaneously, so that any of (14) – (17) can be used in the equalization. If the channel length L_h is known and sample estimates of both $\mathbf{f}_M(n)$ and $\mathbf{b}_M(n)$ are reliable, the symbol estimate can be made more accurate by combining the two. The reliability of the prediction error estimates depends on the magnitude of the channel coefficients $\mathbf{h}(0)$ and $\mathbf{h}(L_h - 1)$. The LSL includes two matrix inversions of size $P \times P$. Algorithms with only one matrix inversion and better numerical stability can be obtained [30]. Algorithms of this type can be derived within the unifying framework of recursive least-squares (RLS) filtering.

At stage m of the lattice, the LSL-algorithm minimizes the trace of the matrix

$$\mathbf{P}_{f,m}(n) = \left[\sum_{i=1}^{n} \lambda^{n-i} \right] \mathbf{f}_m(n) \mathbf{f}_m^H(n) \tag{18}$$

where λ is a forgetting factor (see Appendix A.2). For large n, the summation can be written as

$$\sum_{i=1}^{n} \lambda^{n-i} = \frac{1}{1-\lambda} \tag{19}$$

If (14) or (15) are used, the prediction error covariance matrix estimate must be normalized by $1/(1-\lambda)$ before computing the equalizer coefficients. If $\lambda = 1$, the matrix should be divided by n. The LSL-algorithm provides $\mathbf{P}_{f,M-1}(n)$ and $\mathbf{P}_{b,M-1}(n)$ that are of one order less than the PEF. If M is chosen large enough, the estimates of order $M-1$ are reliable.

3.3 Eigen vector and value tracking

The LSL-algorithm provides the prediction errors and their covariance matrices as needed in (14)–(17). The remaining task is to estimate the channel coefficient $\mathbf{h}(0)$ or $\mathbf{h}(L_h-1)$. A feasible solution is given in [31], where the additive noise is ignored. Using assumption A2, the estimate $\hat{\mathbf{h}}(0)$ of $\mathbf{h}(0)$ can be found as

$$\hat{\mathbf{h}}(0) = \theta_1(n)\mathbf{u}_1(n) \tag{20}$$

where $\mathbf{u}_1(n)$ is the eigenvector corresponding to the largest eigenvalue $\theta_1^2(n)$ of $\mathbf{P}_{f,M}(n)$. The eigenvalue must be normalized by $1/(1-\lambda)$. The estimate $\hat{\mathbf{h}}(L_h-1)$ of $\mathbf{h}(L_h-1)$ can be obtained similarly from $\mathbf{P}_{b,M}(n)$.

The eigen-pair $\theta_1^2(n)$ and $\mathbf{u}_1(n)$ needs to be computed after each update on the prediction error covariance matrix. Adaptive methods to update the eigen vector and value estimates at the arrival of a new data vector are considered in [32]. The complexity of these algorithms increases linearly with P. For small P, no benefit may occur compared to direct methods. In [32], the algorithm of [33] is suggested as a choice for the eigen-pair tracking. It is summarized in Appendix A.3. The algorithm looks complex, because of the singular-value decomposition. However, the number of complex floating point operations required is only on the order of $5P$ per iteration [32].

3.4 The lattice blind equalizer (LBE) algorithm

We are finally ready to present the complete LBE of delay d ($d = 0$ or $d = L_h + M - 1$). The required computations are described below.

Initialize the algorithms of Appendix A.2 and A.3. For every $n = 1, 2, \ldots$

1. Use the algorithm of A.2 to obtain

$$\mathbf{q}(n) = \begin{cases} \mathbf{f}_M(n) & , d = 0 \\ \mathbf{b}_M(n) & , d = L_h + M - 1 \end{cases}$$

and

$$\mathbf{Q}(n) = \begin{cases} \mathbf{P}_{f,M-1}^{-1}(n) & , d = 0 \\ \mathbf{P}_{b,M-1}^{-1}(n) & , d = L_h + M - 1 \end{cases}$$

2. With the algorithm of A.3, compute the dominant eigen-pair $\theta_1^2(n)$ and $\mathbf{u}_1(n)$. Divide $\theta_1^2(n)$ by $1/(1-\lambda)$ or n, if $\lambda = 1$.

3. Compute the channel coefficient estimate $\hat{\mathbf{h}}(d'),\ d' = 0$ for forward prediction or $d' = L_h - 1$ for backward prediction

$$\hat{\mathbf{h}}(d') = \theta_1(n)\mathbf{u}_1(n)$$

4. Use either of the following equations to obtain the MMSE- or ZF-equalizer outputs as

$$\hat{x}^{\mathrm{MMSE}}(n-d) = \sigma_x^2 \hat{\mathbf{h}}^H(d')\mathbf{Q}(n)\mathbf{q}(n) \tag{21}$$

$$\hat{x}^{\mathrm{ZF}}(n-d) = \frac{\hat{\mathbf{h}}^H(d')}{\|\hat{\mathbf{h}}(d')\|^2}\mathbf{q}(n) \tag{22}$$

4 Simulation results

The performance of the lattice blind equalizer (LBE) of either (14) or (15) is evaluated by the following simulations:

A. The convergence of the LBE in white noise for several values of the SNR.

B. The effect of the first channel coefficient magnitude on the equalizer performance.

C. The performance of the $(L_h + M - 1)$-delay LBE resulting from one-step backward prediction.

D. The robustness of the LBE to channel order over-estimation.

E. Convergence of the LBE, when the subchannels have near-common or common zeros.

Zero-delay equalization, obtained with one-step forward prediction is considered in all experiments, except C, where the $(L_h + M - 1)$-delay equalizer resulting from one-step backward prediction is used. The equalizer order is chosen $M = L_h + 2$ for all simulations, except D.

The performance criterion is the sample estimate of the mean-square symbol-error (MSE), defined as

$$\hat{J}_{\mathrm{MSE}} = \frac{1}{N}\Sigma_{n=1}^{N}[|x(n) - \hat{x}(n)|^2] \tag{23}$$

where N is the sample size. The MSE is evaluated after every 50 received symbols by running a test set of 1000 independent symbols through the equalizer. The average of 50 Monte-Carlo runs is presented as the final result.

Subdividing the received signal (2) according to

$$y_i(n) = y_i'(n) + v_i(n) \tag{24}$$

we can define the SNR as in [1]

$$\frac{\mathrm{E}[\Sigma_{j=0}^{P-1}|y_j'(n)|^2]}{\mathrm{E}[\Sigma_{k=0}^{P-1}|v_k(n)|^2]}$$

For all simulations, a 16-QAM signal constellation is used. The additive noise is white and Gaussian. The need to obtain the sample estimate MSE calls for a way to overcome the phase ambiguity inherent in all second-order blind equalization methods. We chose to rotate the estimate $\hat{\mathbf{h}}(d')$ of $\mathbf{h}(d')$ so that the the phases of $\hat{\mathbf{h}}_0(d')$ and $\mathbf{h}_0(d')$ match. Differential coding can be used in applications. The simulated channel is stationary, and therefore $\lambda = 1$ is used. Algorithm initialization parameters are $\delta = 0.1$ and $\rho = 0.01$. The number of samples per symbol is set to $P = 2$.

The magnitude of the impulse response of channels $1 - 3$ used in simulations A–D is given in Fig. 5. The coefficients of each subchannel of channel 1 are presented in Table 1. The magnitude of the impulse response of channels 4–6 used in experiment E is given in Fig. 6 and the zeros of the subchannels of channel 4 in Table 2. Channels 1 and 4 are similar to those used in [25] and [1].

4.1 Convergence in noise

For channel 1, Fig. 7 presents convergence curves for the zero-delay MMSE-LBE (14). Convergence occurs after approximately 500 symbols. Table 3 lists the MSE obtained by an optimum zero-delay equalizer of length $L_g = 7$ and the corresponding MSE obtained by the blind equalizer after 2500 symbols. The LBE clearly approaches the optimum performance. The results are comparable to those obtained using the correlation-based structure of [1], which is not surprising because the two equalizers were shown to be equivalent. Scatter plots of the unequalized and equalized received signal are presented in Fig. 8 for an SNR of 30 dB.

4.2 Effect of the magnitude of the first channel coefficient

We examine the effect of the first channel coefficient magnitude on the equalizer performance. The convergence of the LBE for channel 1 is shown in the top plot of Fig. 9, both when $\mathbf{h}(0)$ is estimated with the eigen-pair tracking algorithm of Appendix A.3 and when it is known exactly. There is no significant difference in the performance, because $\mathbf{h}(0)$ is large compared to the noise.

The MMSE obtained by the optimum zero-delay FSE is higher for channel 2 than for channel 1. The difference between the two curves in the bottom plot of Fig. 9 is also more pronounced, because the magnitude of $\mathbf{h}(0)$ is smaller and its estimation less successful. This can be verified from Fig. 10, where the absolute value of the estimate for channels 1 and 2 is presented.

4.3 LBE with backward prediction

Because the last channel coefficient of channels 1 and 2 is very small, even the optimum $(L_h + M - 1)$-delay FSE fails and one-step backward prediction cannot be used. The last coefficient of channel 3 is large and (15) is now well suited for the equalization. Fig. 11 presents the simulation results, together with the MMSE obtained with the optimum $(L_h + M - 1)$-delay FSE.

4.4 Robustness to channel order over-estimation

One advantage of the prediction-based equalizers is their insensitivity to channel order over-estimation. Repeating experiment A for an SNR of 12.5 dB, we examined the MSE achieved after 2500 symbols for equalizer orders $M = 2, \ldots, 14$. The channel order is $L_h - 1 = 3$. From Fig. 12 we can conclude that the exact channel order is not needed in the LBE.

4.5 Channels with near-common or common subchannel zeros

The existing SOS-based blind equalization/identification methods do not perform well when the subchannels have zeros that are nearly equal [25]. The zeros of the minimum-phase channel 4 are very close to being common, except for one zero-pair, as shown in Table 2. Fig. 13 demonstrates the performance of the LBE; the algorithm converges in spite of the nearly common zeros, albeit more slowly. The subchannel coefficients were equated as $h_1(n) = h_0(n), n = 0, \ldots, L_h - 1$ to construct channel 5. From Fig. 13 we can conclude that the LBE appears to converge even if the subchannels have exactly common zeros.

The zero $(-0.2736 - 0.3719i)$ of both subchannels of channel 5 was changed to $(-0.9500 - 0.8380i)$. The consequent channel 6 is mixed-phase. The results of our final experiment are shown in Fig. 13 and the convergence property still exists. The MSE is high, because zero-delay equalization performs poorly with mixed-phase channels. Consequently, the scatter plots are very unclear and we cannot conclude which channel exactly does the equalizer converge to.

5 Conclusions

We have developed adaptive blind ZF- and MMSE-equalizers based on prediction-error methods and eigen-pair estimation. A significant advantage of the resulting algorithms over previous SOS-based equalizers is their natural recursive structure. Other advantages are their robustness to channel order over-determination and, surprisingly, to channels with common subchannel zeros.

Both forward and backward prediction can be used in the LBE, resulting in zero- and $(L_h + M - 1)$-delay equalization, respectively. Here lies also the weakness of the approach; equalizers of arbitrary delay cannot be readily obtained. This problem is the subject of on going work, with a possible solution lying in modifying the LSL-algorithm to perform multi-step forward prediction and using the methods presented in [20]. The robustness to channels with common subchannel zeros also needs to be investigated further.

A Appendix

A.1 Proof of theorems

Proof of Theorem 1: We begin our proof from the correlation matrix -based blind zero-delay MMSE-equalizer $\mathbf{g}_0^{\mathrm{MMSE}}$ given in [1]. We have

$$\mathbf{g}_0^{\mathrm{MMSE}}\mathbf{R}_{M+1} = \sigma_x^2[\mathbf{h}^H(0),\ \mathbf{0}_{1\times MP}] \tag{25}$$

which can be divided into two parts as

$$\mathbf{g}_0^{\mathrm{MMSE}}\mathbf{r}_P^f = \sigma_x^2\mathbf{h}^H(0) \tag{26}$$

$$\mathbf{g}_0^{\mathrm{MMSE}}\mathbf{r}_{MP}^f = \mathbf{0}_{1\times MP} \tag{27}$$

where $\mathbf{r}_P^f$ contains the first P and $\mathbf{r}_{MP}^f$ the last MP columns of $\mathbf{R}_{M+1}$. From (13), the forward prediction error covariance is $\mathbf{P}_{f,M} = \mathbf{A}_M\mathbf{r}_P^f$. We can write (26) as

$$\mathbf{g}_0^{\mathrm{MMSE}}\mathbf{r}_P^f = \sigma_x^2\mathbf{h}^H(0)\mathbf{P}_{f,M}^{-1}\mathbf{P}_{f,M} \tag{28}$$

$$\mathbf{g}_0^{\mathrm{MMSE}} = \sigma_x^2\mathbf{h}^H(0)\mathbf{P}_{f,M}^{-1}\mathbf{A}_M \tag{29}$$

which is the desired result (14). Furthermore, (27) holds after substituting for $\mathbf{g}_0^{\mathrm{MMSE}}$ from (29) and noting from (13) that $\mathbf{A}_M\mathbf{r}_{MP}^f = \mathbf{0}_{P\times MP}$.

It is straightforward to modify the proof to obtain (15) for one-step backward linear prediction. We have [1]

$$\mathbf{g}_{L_h+M-1}^{\mathrm{MMSE}}\mathbf{r}_P^b = \sigma_x^2\mathbf{h}^H(L_h-1) \tag{30}$$

$$\mathbf{g}_{L_h+M-1}^{\mathrm{MMSE}}\mathbf{r}_{MP}^b = \mathbf{0}_{1\times MP} \tag{31}$$

where $\mathbf{r}_P^b$ consists of the last P and $\mathbf{r}_{MP}^b$ the first MP columns of $\mathbf{R}_{M+1}$. From (13), the backward prediction error covariance $\mathbf{P}_{b,M} = \mathbf{B}_M\mathbf{r}_P^b$, and

$$\mathbf{g}_{L_h+M-1}^{\mathrm{MMSE}} = \sigma_x^2\mathbf{h}^H(L_h-1)\mathbf{P}_{b,M}^{-1}\mathbf{B}_M \tag{32}$$

which is (15). Equation (31) holds after substituting for $\mathbf{g}_{L_h+M-1}^{\mathrm{MMSE}}$ from (32) and noting from (13) that $\mathbf{B}_M\mathbf{r}_{MP}^b = \mathbf{0}_{P\times MP}$. The proof verifies the equivalence of the prediction-based and correlation matrix based MMSE-equalizers. □

Proof of Theorem 2: When the forward prediction-error filter is optimum in the mean-square sense, the PEF input $\mathbf{y}_M(n-1)$ and the prediction error $\mathbf{f}_M(n)$ are orthogonal [27]. Therefore

$$\mathrm{E}[\mathbf{y}_M(n-1)\mathbf{f}_M^H(n)] = \mathbf{0}_{MP\times P} \tag{33}$$

Following the proof given in [31], we substitute for $\mathbf{f}_M^H(n)$ from (11), use assumption A2 and ignore the noise to obtain

$$\mathbf{f}_M(n) = \mathbf{h}(0)x(n) \tag{34}$$

and consequently (16). The proof of (17) follows in a similar way after recognizing that the orthogonality condition for backward prediction is

$$\mathrm{E}[\mathbf{y}_M(n)\mathbf{b}_M^H(n)] = \mathbf{0}_{MP\times P} \tag{35}$$

We show that the prediction-based ZF-equalizers (16) and (17) are equivalent to the correlation matrix based equalizers given in [1]. Forward prediction and (16) is considered first. In [1], the ZF-equalizer is obtained from the MMSE-equalizer (25) when the noise power is assumed zero. We need to establish a similar link between the prediction-based ZF- and MMSE-equalizers (16) and (14). When $\sigma_v^2 = 0$, the covariance matrix $\mathbf{P}_{f,M}$ of the prediction error $\mathbf{f}_M$ can be obtained from (34) as

$$\mathbf{P}_{f,M} = \sigma_x^2\mathbf{h}(0)\mathbf{h}^H(0) \tag{36}$$

Substituting (36) into (14), it is straightforward to show that (16) follows, since $\mathbf{g}^{\mathrm{ZF}} = \mathbf{g}^{\mathrm{MMSE}}$ in the absence of noise. With obvious changes, the $(L_h + M - 1)$-delay equalizer (17) results from (15) when $\sigma_v^2 = 0$. □

A.2 The LSL-algorithm

The following variables (cf. Fig. 4) are involved in the LSL-algorithm at prediction order m and time instant n:

$\mathbf{f}_m(n)$	forward prediction error vector
$\mathbf{b}_m(n)$	backward prediction error vector
$\mathbf{P}_{f,m}(n)$	covariance matrix of $\mathbf{f}_m(n)$
$\mathbf{P}_{b,m}(n)$	covariance matrix of $\mathbf{b}_m(n)$
$\kappa_{f,m}(n)$	forward reflection coefficient matrix
$\kappa_{b,m}(n)$	backward reflection coefficient matrix
$\Delta_m(n)$	cross-covariance matrix between $\mathbf{f}_m(n)$ and $\mathbf{b}_m(n)$
$\gamma_m(n)$	conversion factor

The first two variables are of dimension $1 \times P$, the five next ones $P \times P$ and the last one is a scalar. To accommodate time variations in the data, a forgetting factor λ is introduced. It takes on values between zero and one. Prewindowing is assumed, so that $\mathbf{y}(n) = 0,\ n \leq 0$. The LSL-algorithm is described below.

Initialize the algorithm for $n = 0$

$$\begin{aligned}\mathbf{b}_{m-1}(0) &= \mathbf{0}\\ \Delta_{m-1}(0) &= \mathbf{0}\\ \mathbf{P}_{f,m-1}(0) &= \mathbf{P}_{b,m-1}(-1) = \delta\mathbf{I}\end{aligned}$$

where $m = 1, \ldots, M$ and δ is a small positive constant. For every received symbol $\mathbf{y}(n), n = 1, 2, \ldots,$ initialize the zeroth-order variables

$$\begin{aligned}\mathbf{f}_0(n) &= \mathbf{b}_0(n) = \mathbf{y}(n)\\ \gamma_0(n-1) &= 1\end{aligned}$$

and find the mth-order ($m = 1, \ldots, M$) forward and backward prediction errors iteratively as follows:

1. Update the covariance matrices

$$\begin{aligned}\Delta_{m-1}(n) =& \lambda\Delta_{m-1}(n-1)\\ &+\mathbf{f}_{m-1}(n)\mathbf{b}_{m-1}^H(n-1)/\gamma_{m-1}(n-1)\\ \mathbf{P}_{b,m-1}(n-1) =& \lambda\mathbf{P}_{b,m-1}(n-2)\\ &+\mathbf{b}_{m-1}(n-1)\mathbf{b}_{m-1}^H(n-1)/\gamma_{m-1}(n-1)\\ \mathbf{P}_{f,m-1}(n) =& \lambda\mathbf{P}_{f,m-1}(n-1)\\ &+\mathbf{f}_{m-1}(n)\mathbf{f}_{m-1}^H(n)/\gamma_{m-1}(n-1)\end{aligned}$$

2. Compute the reflection coefficient matrices and update the conversion factor

$$\begin{aligned}\kappa_{f,m}(n) =& -\Delta_{m-1}(n)\mathbf{P}_{b,m-1}^{-1}(n-1)\\ \kappa_{b,m}(n) =& -\Delta_{m-1}^H(n)\mathbf{P}_{f,m-1}^{-1}(n)\\ \gamma_m(n-1) =& \gamma_{m-1}(n-1)\\ &-\mathbf{b}_{m-1}^H(n-1)\mathbf{P}_{b,m-1}^{-1}(n-1)\mathbf{b}_{m-1}(n-1)\end{aligned}$$

3. Obtain the forward and backward prediction error vectors of order m

$$\mathbf{f}_m(n) = \mathbf{f}_{m-1}(n) + \kappa_{f,m}(n)\mathbf{b}_{m-1}(n-1)$$
$$\mathbf{b}_m(n) = \mathbf{b}_{m-1}(n-1) + \kappa_{b,m}^H(n)\mathbf{f}_{m-1}(n)$$

4. If $m < M$ return to step 1, otherwise terminate computation.

A.3 Adaptive eigen-pair tracking

The dominant eigen-pair can be updated at time n after the following parameters from the LSL-algorithm have become available:

$\mathbf{q}(n)$	prediction error $\mathbf{f}_M(n)$ or $\mathbf{b}_M(n)$
$\gamma_M(n-1)$	conversion factor
λ	forgetting factor, $0 \leq \lambda \leq 1$

The parameters of interest are:

$\theta_1^2(n)$	the largest eigenvalue
$\mathbf{u}_1(n)$	eigenvector corresponding to $\theta_1^2(n)$

Initialize the algorithm at time instant $n = 0$:

$$p(0) = \theta_1(0) = 1$$
$$\mathbf{u}_1(0) = \rho[1, \ldots, 1]^T$$

where ρ is a small number. For every $n = 1, 2, \ldots$

1. Compute the variables

$$z = \mathbf{u}_1^H(n-1)\mathbf{q}(n)$$
$$\omega = \mathbf{q}(n) - z\mathbf{u}_1(n-1)$$
$$c = \|\omega\| \ , \quad \omega = \omega/c$$

2. Let $\alpha \triangleq \sqrt{\gamma_M(n-1)}$ and form the matrix

$$\mathbf{B} = \begin{bmatrix} \sqrt{\lambda}\theta_1(n-1) & 0 & z/\alpha(n) \\ 0 & \sqrt{\lambda p(n-1)} & c/\alpha(n) \end{bmatrix}$$

3. Compute the singular-value decomposition

$$\mathbf{X}\Sigma\mathbf{Y}^H = \mathbf{B}$$

4. The square-root $\theta_1(n)$ of the largest eigen-value is the top left-hand element of Σ and the corresponding eigen-vector $\mathbf{u}_1(n)$ is the first column of

$$\left[\mathbf{u}_1(n-1), \quad \omega\right]\mathbf{X}$$

5. Let $\theta_2(n)$ be the second element of the second row of Σ. Update the parameter $p(n)$, needed at step 2 in the next iteration, according to

$$p(n) = \theta_2^2(n) + (P-2)\lambda p(n-1)$$

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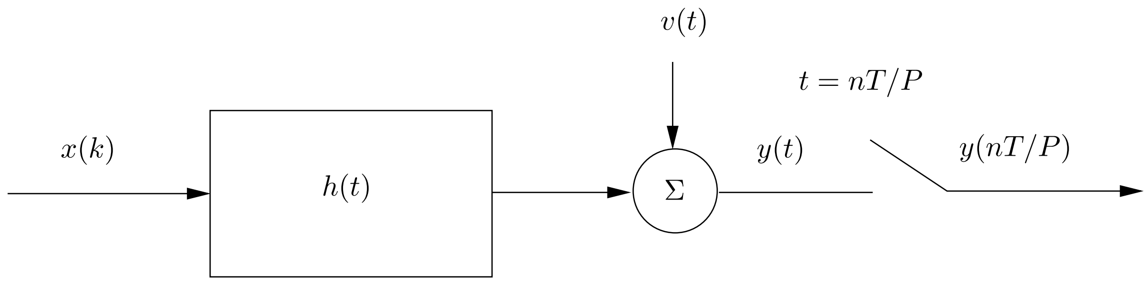


Figure 1: Communication system model.

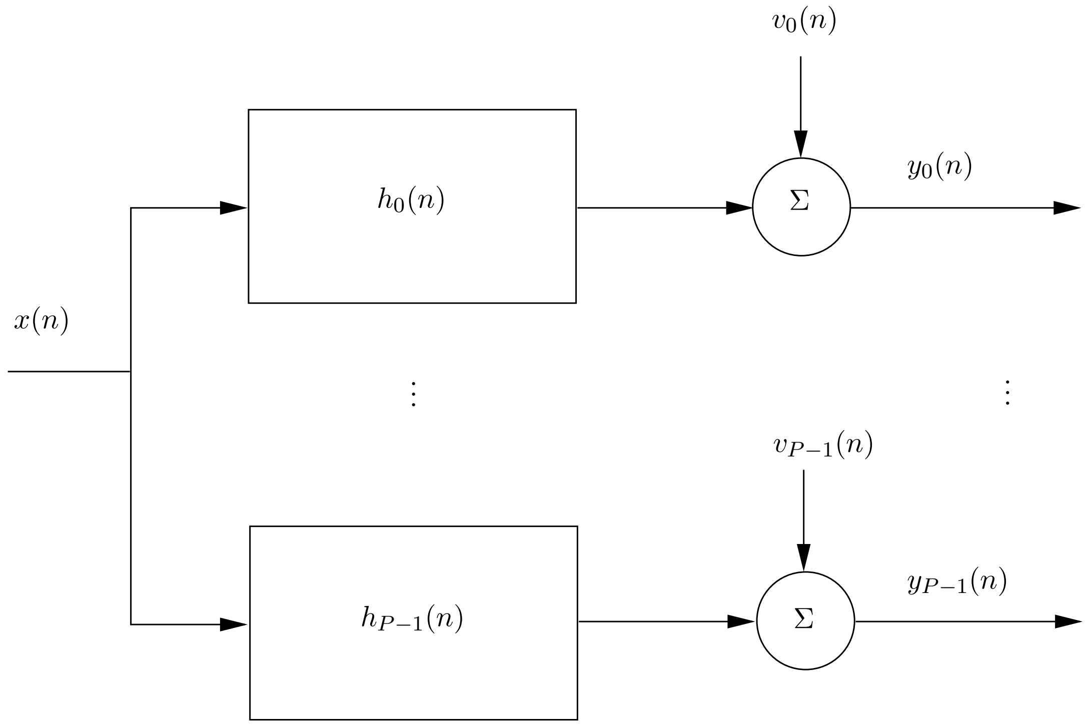


Figure 2: SIMO communication system model.

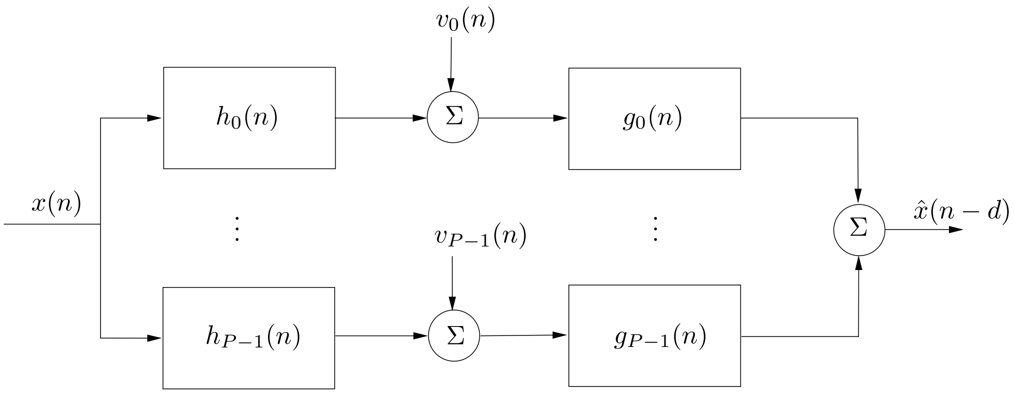


Figure 3: Equalization of a SIMO channel.

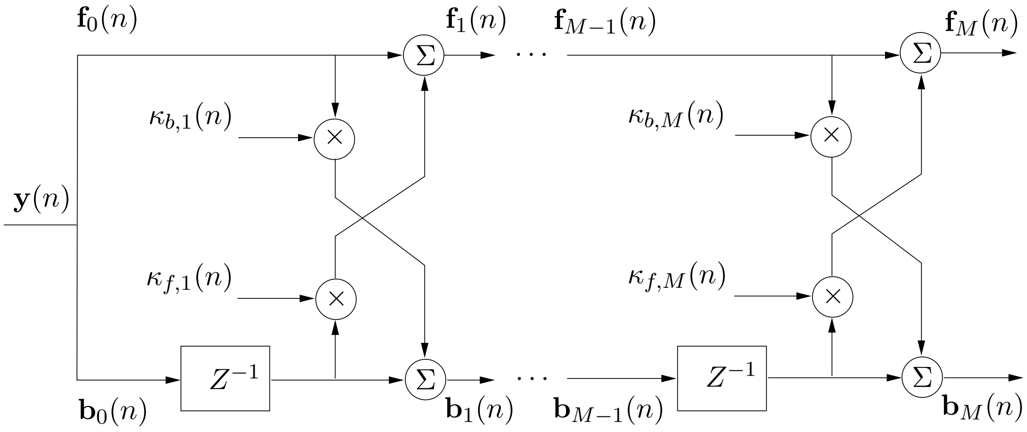


Figure 4: Multistage lattice predictor.

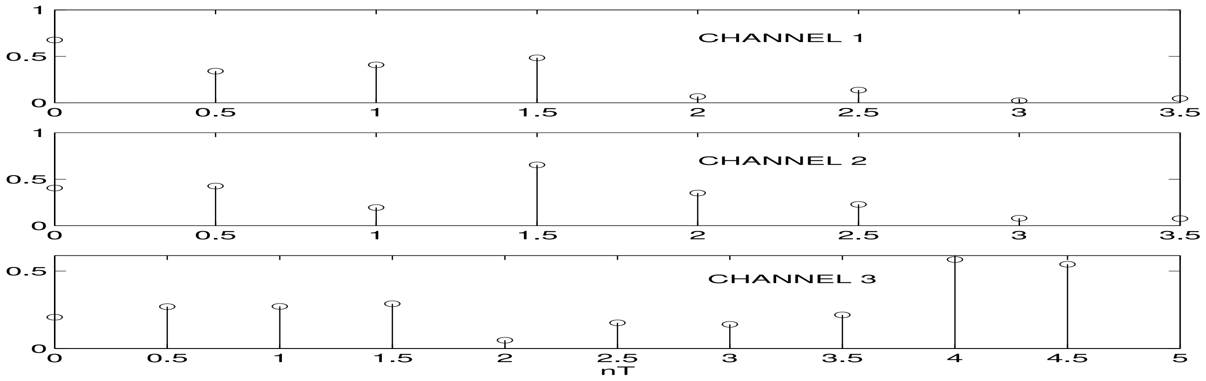


Figure 5: Magnitude of the channel impulse responses used in experiments A–D.

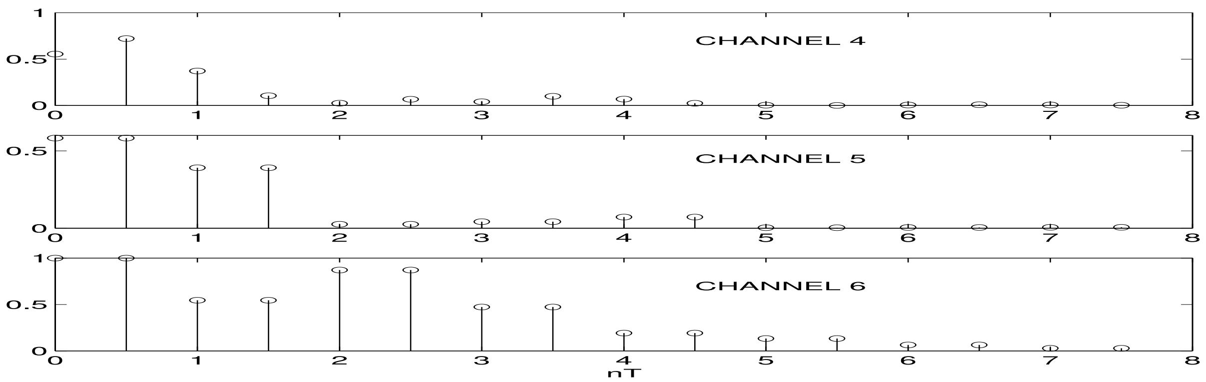


Figure 6: Magnitude of the channel impulse responses used in experiment E.

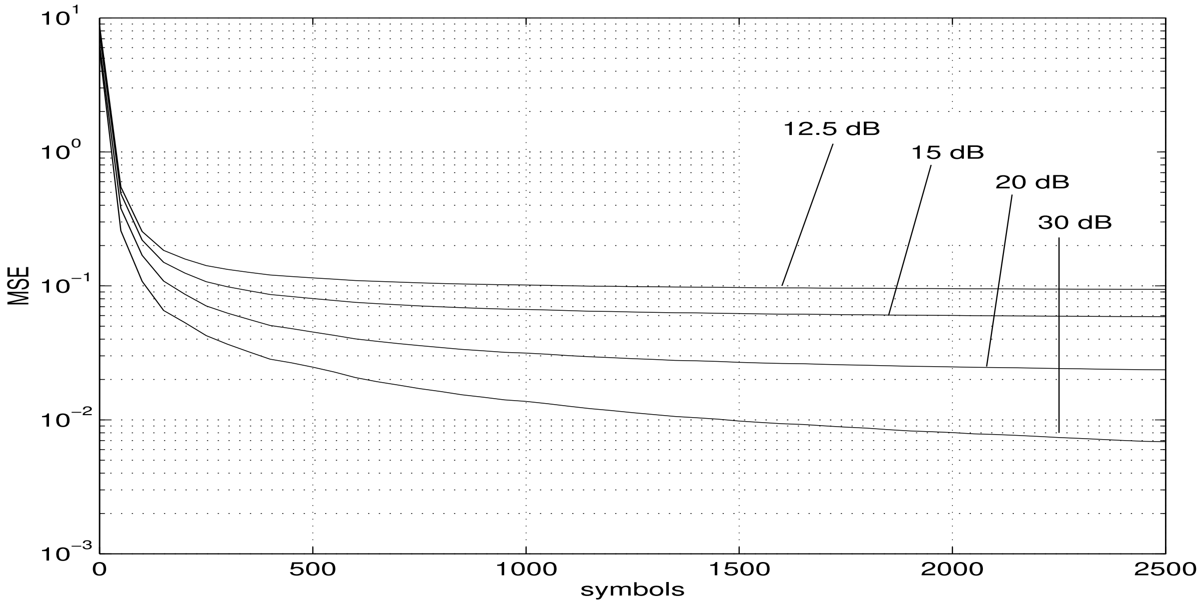
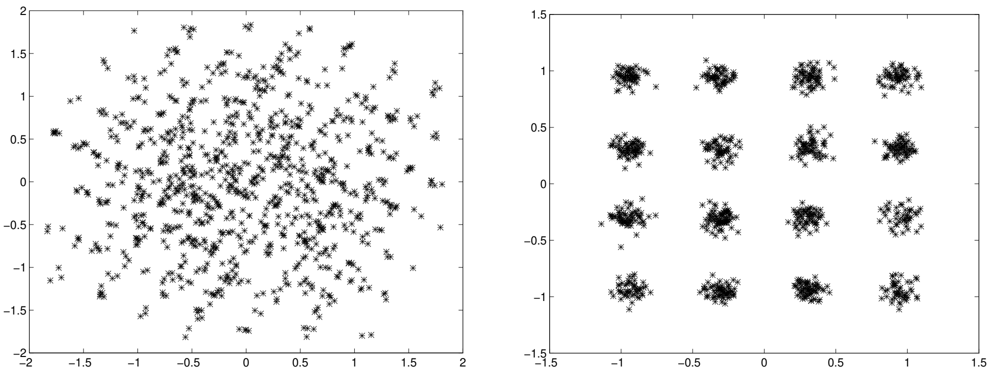


Figure 7: MSE of the zero-delay LBE for different SNRs.



(a) Unequalized

(b) Equalized

Figure 8: Scatter plots before and after equalization, SNR 30 dB.

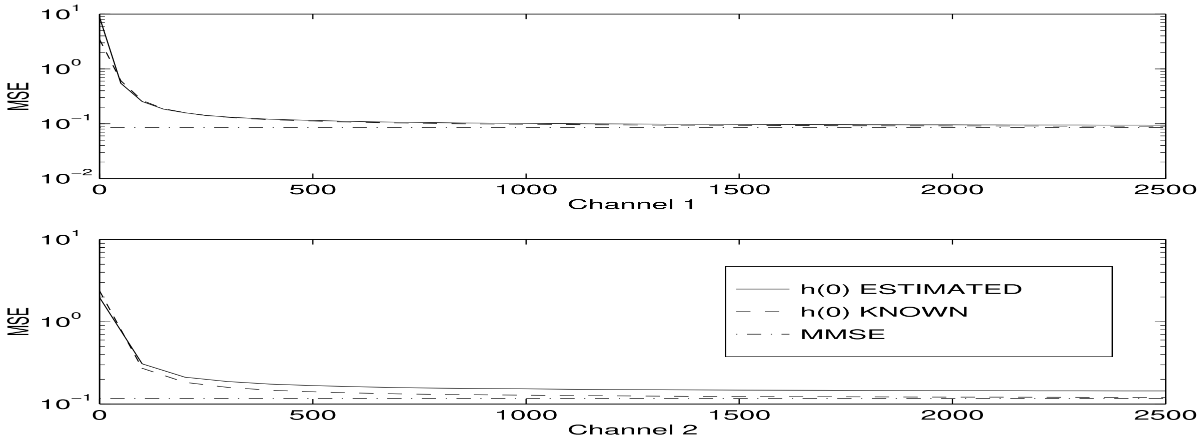


Figure 9: LBE performance when $\mathbf{h}(0)$ is estimated and when it is known, channels 1 and 2, SNR 12.5 dB.

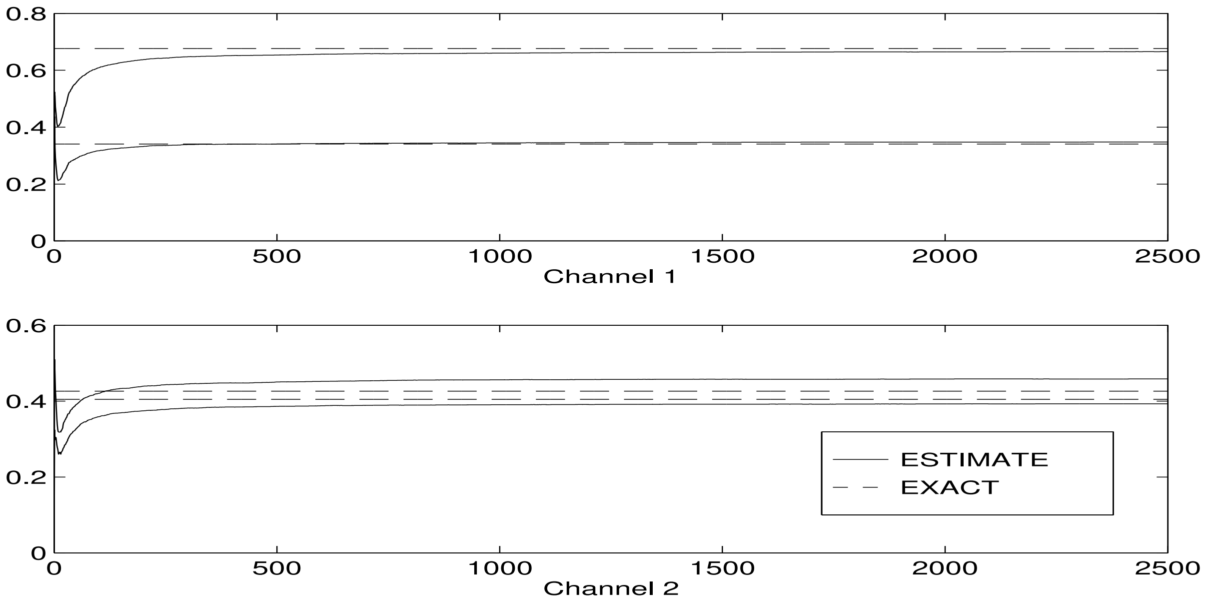


Figure 10: Magnitude of the estimate and exact value of the channel coefficients $h_0(0)$ and $h_1(0)$.

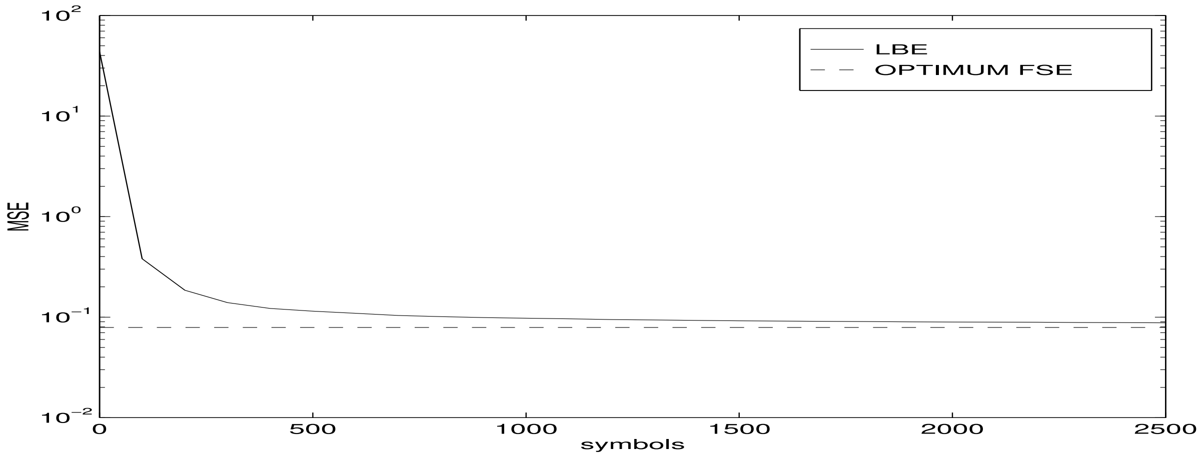


Figure 11: LBE performance with backward prediction, channel 3, SNR 12.5 dB.

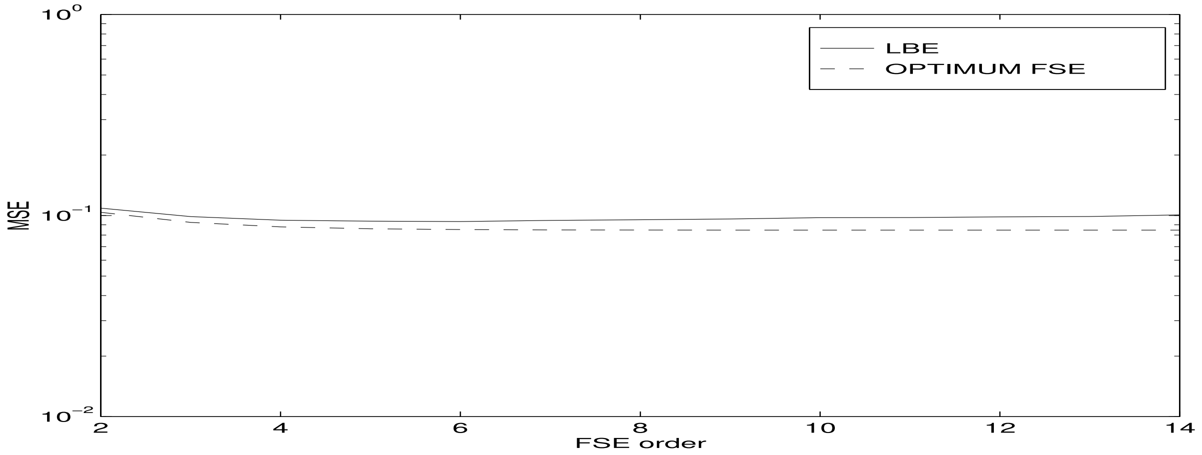


Figure 12: LBE and optimum FSE performance after 2500 symbols for several equalizer orders.

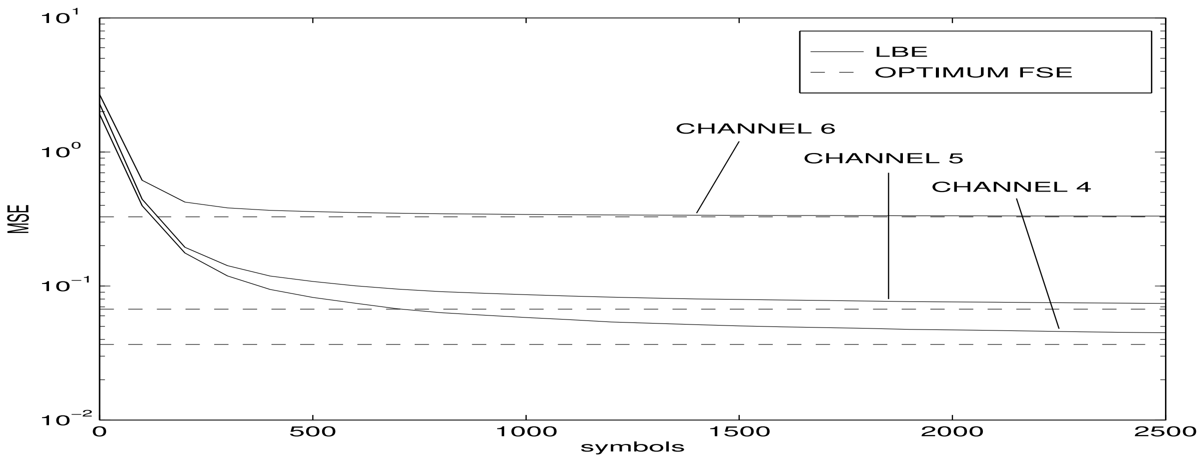


Figure 13: LBE performance for channels 4–6 with near-common or common subchannel zeros, SNR 12.5 dB.

Table 1: Channel 1 coefficients

Subchannel coefficients	
$h_0(n)$	$h_1(n)$
$0.3976 - 0.5472i$	$0.0946 - 0.3271i$
$-0.3647 + 0.1837i$	$-0.3696 + 0.3119i$
$-0.0396 + 0.0544i$	$0.1032 - 0.0903i$
$0.0134 - 0.0184i$	$-0.0346 + 0.0295i$

Table 2: Channel 4 zeros

Subchannel zeros	
$h_0(n)$	$h_1(n)$
$-0.2736 - 0.3719i$	$-0.2730 - 0.3716i$
$0.0074 - 0.4999i$	$0.0086 - 0.4981i$
$-0.4560 + 0.2706i$	$-0.4559 + 0.2715i$
$-0.2472 + 0.3870i$	$-0.2468 + 0.3856i$
$0.5213 - 0.0539i$	$0.5206 - 0.0546i$
$0.4149 + 0.1422i$	$0.4150 + 0.1416i$
$0.1961 + 0.7765i$	$0.1008 + 0.2539i$

Table 3: LBE performance compared to the optimum MMSE

	MSE	
SNR [dB]	Optimum FSE	Forward LBE
12.5	0.0852	0.0942
15	0.0514	0.0589
20	0.0177	0.0236
30	0.0019	0.0069